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To Thomas Stewart Wesner

Dad

To Margot

Phil

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C H A P T E R

5 The Trigonometric Functions

In this and the following three chapters we will study trigonometry. The word trigonometry means "triangle measurement."[1] Trigonometry was used by the ancient Greeks to study astronomy, and was well developed by the eighteenth century. By then it was also used in surveying and in the physics developed by Sir Isaac Newton and others. In the age of the computer, trigonometry has become more important than ever. Computer robots are programmed using trigonometric descriptions of their motions, computer graphics use trigonometry extensively to generate screen images, and voice recognition by computers uses Fourier transforms, a concept built on the trigonometric functions.

5–1 The trigonometric ratios

Find the speed relative to the ground of a boat heading straight across a river at 16 knots if the current is moving at 4.3 knots.

The mathematics we study in this section can be used to solve this problem. It requires some basic physics and a law that has been known for well over 2,000 years.

Degree measure for angles

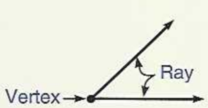


Figure 5–1

An **angle** is composed of two rays, both beginning at what is called the **vertex** of the angle. Figure 5–1 shows a representation of a ray and of an angle.

Angles are often measured in degrees. The notation for degrees is °. A common and useful interpretation of angle measure is "the amount of rotation" of one ray away from the other. In this context 90° corresponds to a

[1]The word *trigonometry* is credited to Bartholomaus Pitiscus (1561–1613).

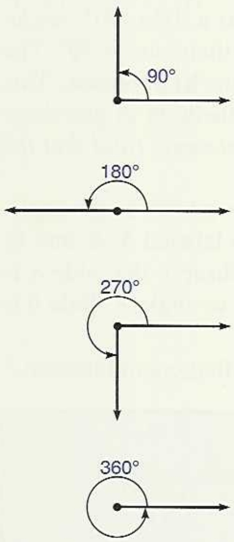


Figure 5–2

quarter-rotation, 180° to a half-rotation, 270° to three-quarters of a rotation, and 360° to a full rotation. See figure 5–2. An angle with measure between 0° and 90° is said to be **acute;** an angle with measure 90° is **right** and an angle with measure between 90° and 180° is **obtuse.**

There are two systems in wide use for measuring angles in degrees: the **degree, minute, second (DMS) system** and the **decimal degree system.**

In the first system, parts of an angle are broken down into 60ths, called minutes, and minutes are broken down into 60ths, called seconds. This means there are $60^2 = 3{,}600$ seconds in one degree. The symbols °, ′, and ″ are used for degrees (°), minutes (′), and seconds (″), respectively. It is much like our system for keeping time. For example, 16°4′22″ means 16 degrees, 4 minutes, 22 seconds. Observe that the 4 minutes is $\frac{4}{60}$ degree and that 22 seconds is $\frac{22}{60^2}$ degree. We use this idea in example 5–1 A. Calculations are easiest in decimal degrees, and so it is important to be able to convert to this form if necessary. Many electronic calculators have special keys to convert angles in degrees, minutes, and seconds to decimal form. This is illustrated in example 5–1 A for two typical calculators, as well as for the TI-81 graphing calculator, which does not have special keys for this purpose.

■ ***Example 5–1 A***

Convert 12°20′34″ to decimal form. Round to the nearest 0.001°.

$12 + \dfrac{20}{60} + \dfrac{34}{60^2}$ degrees — Rewrite minutes and seconds as fractional parts of a degree

12.3427777 . . . ° — Compute the decimal form of the fractions

12.343° — Round to the nearest 0.001°

Typical key sequences:

Calculator 1: 12 $\boxed{\circ\ \prime\ \prime\prime}$ 20 $\boxed{\circ\ \prime\ \prime\prime}$ 34 $\boxed{\circ\ \prime\ \prime\prime}$

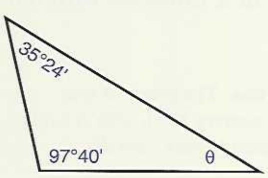
Calculator 2: 12.2034 $\boxed{\to\text{H}}$ "H" stands for Hours.

TI-81: 12 $\boxed{+}$ 20 $\boxed{\div}$ 60 $\boxed{+}$ 34 $\boxed{\div}$ 60 $\boxed{x^2}$ $\boxed{\text{ENTER}}$ ■

Several properties of triangles

A **triangle** is a closed figure of three sides and three angles. It is a theorem that *the sum of the measures of the angles of a triangle is 180°.* We often denote angles using Greek letters. The letters most often used are θ (theta), α (alpha), and β (beta). If we know the measure of two of the angles in a triangle we can compute the measure of the third by subtraction.

■ ***Example 5–1 B***



Find the measure of angle θ in the triangle.

$$
\begin{aligned}
35°24' + 97°40' &= 132°64' \\
&= 133°4' && 64' = 1°4' \\
180° - 133°4' &= 179°60' && 1° = 60' \\
&\quad \underline{-\ 133°4'} \\
&\quad\ \ 46°56'
\end{aligned}
$$

Thus the measure of θ is 46°56′. ■

A **right triangle** is a triangle in which one of the angles is a right (90°) angle. In such a triangle, two of the angles must be acute, since their sum is 90°. The side of a right triangle opposite the right angle is called the **hypotenuse.** This is always the longest side of the triangle because it is a theorem of geometry that *the longest side of any triangle is opposite the largest angle (and that the shortest side is opposite the smallest angle).*

It is important to recognize which side is opposite which acute angle. Figure 5–3 shows a right triangle in which the angles are labeled *A, B,* and *C,* and the sides are *a, b,* and *c.* Angle *A* is highlighted. Observe that side *a* is opposite to angle *A* and that side *b* is said to be adjacent to angle *A*. Side *b* is opposite angle *B,* and side *a* is adjacent to angle *B*.

One of the most-used facts in mathematics is the Pythagorean theorem.[2]

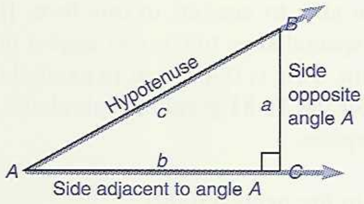


Figure 5–3

The Pythagorean theorem

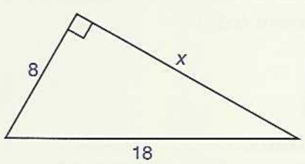
In a right triangle with hypotenuse of length *c* and sides of lengths *a* and *b,*

$$a^2 + b^2 = c^2$$

This theorem allows us to find the measure of any side of a right triangle when we know the measure of the other two sides.

■ *Example 5–1 C*

Find the length of side x in the triangle.



Using the Pythagorean theorem we obtain

$$x^2 + 8^2 = 18^2$$
$$x^2 + 64 = 324$$
$$x^2 = 260$$
$$x = \sqrt{260}$$
$$x = 2\sqrt{65} \approx 16.1 \qquad \sqrt{260} = \sqrt{4 \cdot 65} = \sqrt{4} \cdot \sqrt{65}$$

■

A physical law

As we noted earlier the Pythagorean theorem has many applications. One applies to a fact of physics, concerning the flight of an aircraft through the air.[3] The **airspeed** of an aircraft is its speed relative to the air. It is the speed that the instruments in the aircraft actually measure. It is generally *not* the speed of the aircraft as measured from the ground. The **heading** of an aircraft is the direction in which it is pointed. The aircraft does not actually travel in this direction unless it is pointed directly into the wind or is pointed with the wind. Usually the wind causes the aircraft to travel in a direction different than that in which it is pointed. See figure 5–4.

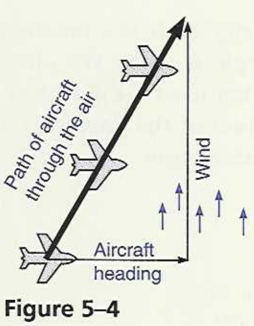


Figure 5–4

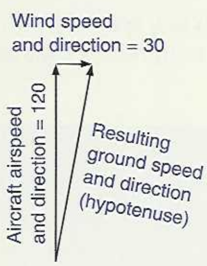
[2]The word *theorem* means a statement that has been proved to be true. The proof of this theorem is credited to the Greek mathematician Pythagoras (sixth century B.C.), who is said to have sacrificed an ox as an offering of thanks. In the last two thousand years, literally hundreds of proofs of this theorem have been given.

[3]We are describing a specific application of vectors. The topic is discussed further in chapter 8.

If the direction of the wind and the heading of an aircraft are perpendicular, then they can be described by the sides of a right triangle. The lengths of the sides describe the airspeed of the aircraft and the speed of the wind. We show the sides as arrows, called vectors, to indicate the direction of the wind and the direction in which the aircraft is pointed. The vector (arrow) for the wind begins at the end of the vector for the aircraft.

The length of the hypotenuse represents the speed of the aircraft relative to the ground, called its **ground speed.** The hypotenuse is drawn as a vector also, beginning at the beginning of the vector for the aircraft, and ending at the end of the vector for the wind. The direction of this vector represents the direction in which the aircraft actually moves, relative to the ground.

■ *Example 5–1 D*



An aircraft is flying with an airspeed of 120 knots[4] and a heading of due north. A 30 knot wind is blowing from the west. What is the aircraft's ground speed v, to the nearest knot?

Represent the speeds and directions given above in the right triangle as shown. The ground speed v is the length of the hypotenuse.

$$v^2 = 120^2 + 30^2$$
$$v^2 = 15{,}300$$
$$v = \sqrt{15{,}300} \approx 123.69$$

Thus, the ground speed is about 124 knots. ■

The trigonometric ratios

In the fifteenth and sixteenth centuries in Germany, trigonometry developed into the form we now present. We begin with a definition of three **trigonometric ratios.** These are, along with their abbreviations, **sine** (sin), **cosine** (cos), and **tangent** (tan). Later we will define three more ratios.

The primary trigonometic ratios

If θ is either of the two acute angles in a right triangle, then

Ratio **Definition**

$$\sin\theta = \frac{\text{length of side opposite } \theta}{\text{length of hypotenuse}}$$

$$\cos\theta = \frac{\text{length of side adjacent to } \theta}{\text{length of hypotenuse}}$$

$$\tan\theta = \frac{\text{length of side opposite to } \theta}{\text{length of side adjacent to } \theta}$$

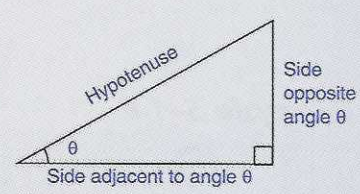
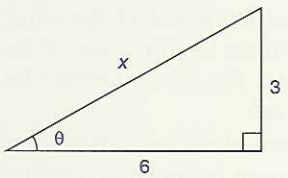


Figure 5–5

The first ratio means "the sine of angle θ" and is usually read "sine theta." The other two ratios are interpreted similarly. These ratios are used in astronomy, surveying, engineering, science, and mathematics—there is in fact virtually no area of science and technology that does not use them somewhere. Example 5–1 E illustrates the application of these definitions to a particular triangle.

[4]One knot means 1 nautical mile ($\approx$ 6,080.27 ft) per hour. Recall that 1 mile is 5,280 feet.

■ *Example 5–1 E*



1. Find the sine, cosine, and tangent ratios for angle θ.

$$x^2 = 3^2 + 6^2$$

Use the Pythagorean theorem to find x

$$x^2 = 45$$

$$x = \sqrt{45} = \sqrt{9 \cdot 5} = 3\sqrt{5}$$

$$\sin\theta = \frac{\text{length of side opposite }\theta}{\text{length of hypotenuse}} = \frac{3}{3\sqrt{5}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5} \qquad \frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

$$\cos\theta = \frac{\text{length of side adjacent to }\theta}{\text{length of hypotenuse}} = \frac{6}{3\sqrt{5}} = \frac{2\sqrt{5}}{5} \qquad \frac{\overset{2}{\cancel{6}}}{\underset{1}{\cancel{3}}\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

$$\tan\theta = \frac{\text{length of side opposite }\theta}{\text{length of side adjacent to }\theta} = \frac{3}{6} = \frac{1}{2}$$

■

The final three trigonometric ratios are called the **cosecant** (csc), **secant** (sec), and **cotangent** (cot). These ratios can be defined as the reciprocals of the primary ratios.

> **Reciprocal pair ratios**
>
> If θ is either acute angle of a right triangle, then
>
> $$\csc\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta}$$

It is worth stressing that the following pairs of ratios are reciprocals:

sine and cosecant
cosine and secant
tangent and cotangent

The definitions above also imply the following relations:

$$\sin\theta = \frac{1}{\csc\theta}, \quad \cos\theta = \frac{1}{\sec\theta}, \quad \tan\theta = \frac{1}{\cot\theta}$$

These relations and the definitions mean that if we know one ratio we can invert it to find another ratio. The example 5–1 F illustrates this.

■ *Example 5–1 F*

Find the value of the other member of the appropriate reciprocal pair.

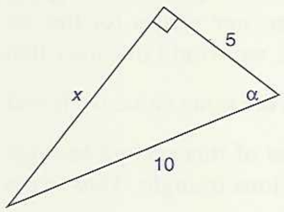
1. $\sin\alpha = \frac{2}{3}$ $\qquad \csc\alpha = \frac{3}{2}$, since $\frac{1}{\frac{2}{3}} = \frac{3}{2}$

2. $\sec\alpha = 1.6$ $\qquad \cos\alpha = \frac{1}{1.6} = \frac{1}{1.6} \cdot \frac{10}{10} = \frac{10}{16} = \frac{5}{8}$

■

Example 5–1 G applies the idea of reciprocal functions to a particular triangle.

Example 5–1 G



Find the value of the six trigonometric ratios for angle α in the triangle.

$$10^2 = x^2 + 5^2 \quad \text{Use the Pythagorean theorem to find } x$$
$$75 = x^2$$
$$5\sqrt{3} = x \quad \sqrt{75} = \sqrt{25 \cdot 3} = 5\sqrt{3}$$

$$\sin \alpha = \frac{x}{10} = \frac{5\sqrt{3}}{10} = \frac{\sqrt{3}}{2}, \ \csc \alpha = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}, \quad \csc \alpha = \frac{1}{\sin \alpha}$$

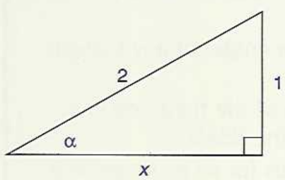
$$\cos \alpha = \frac{5}{10} = \frac{1}{2}, \quad \sec \alpha = 2, \quad \sec \alpha = \frac{1}{\cos \alpha}$$

$$\tan \alpha = \frac{x}{5} = \frac{5\sqrt{3}}{5} = \sqrt{3}, \ \cot \alpha = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}, \quad \cot \alpha = \frac{1}{\tan \alpha}$$

If we know one of the trigonometric ratios of an acute angle, we can construct a right triangle with an angle for which that ratio is true. We can use this triangle to compute the other five trigonometric ratios.

Example 5–1 H

Solve each problem.



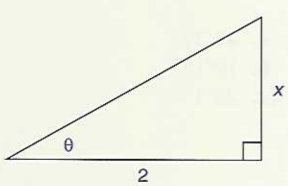
1. In a right triangle with acute angle α, $\csc \alpha = 2$. Construct a right triangle for which this is true, and use the triangle to find the values of the other five trigonometric ratios for angle α.

Since $\csc \alpha = 2$, $\sin \alpha = \frac{1}{2}$. Since $\sin \alpha = \dfrac{\text{length of side opposite } \alpha}{\text{length of hypotenuse}} = \frac{1}{2}$, we can use a triangle in which the hypotenuse has length 2 and the length of the side opposite angle α is 1, as shown in the figure.

$$2^2 = x^2 + 1^2 \quad \text{Find the value of side } x$$
$$3 = x^2$$
$$\sqrt{3} = x$$

$$\cos \alpha = \frac{x}{2} = \frac{\sqrt{3}}{2}, \qquad \sec \alpha = \frac{1}{\cos \alpha} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3},$$

$$\tan \alpha = \frac{1}{x} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}, \ \cot \alpha = \frac{1}{\tan \alpha} = \frac{\sqrt{3}}{1} = \sqrt{3}$$



2. Suppose $\tan \theta = \dfrac{x}{2}$ and $x > 0$. Construct a triangle for which this is true, and use it to compute $\cos \theta$.

$\tan \theta = \dfrac{x}{2} = \dfrac{\text{opp}}{\text{adj}}$. The triangle shown meets this description. The length of the hypotenuse is $\sqrt{x^2 + 2^2} = \sqrt{x^2 + 4}$.

$$\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{2}{\sqrt{x^2 + 4}}.$$

Note As shown in part 1 of example 5–1 H, *when given the value of one of the reciprocal trigonometric ratios, it is best to immediately convert that value to a primary trigonometric ratio.*

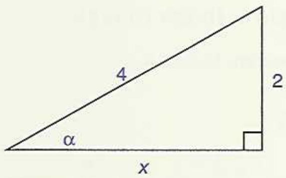


Figure 5–6

A logical question is whether any other triangle could be used in part 1 of example 5–1 H. The answer is yes. For example, since $\frac{2}{4}$ reduces to $\frac{1}{2}$ we could use the triangle shown in figure 5–6. However, our values for the six trigonometric ratios would not change. For example, we would discover that $x = 2\sqrt{3}$, so that $\cos \alpha = \frac{x}{4} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$, which is the same value we found in the example. In fact, the lengths of the three sides of this second triangle are each double the corresponding length in the previous triangle. Thus, each ratio would reduce to the previous values.

Similarly we could start with any fraction equivalent to $\frac{1}{2}$. This means that we could use an unlimited number of triangles in such problems and always arrive at the same values of the six trigonometric ratios.

Mastery points

Can you

- State whether a given angle is acute, right, or obtuse?
- Convert an angle given in degrees, minutes, and seconds into decimal form?
- Use the fact that the sum of the measures of the angles of any triangle is 180° to determine unknown angles?
- Use the Pythagorean theorem to find the length of the third side in a right triangle when given the lengths of two of the sides?
- State the definitions of the six trigonometric ratios for an acute angle θ in a right triangle?
- Find the values of the six trigonometric ratios for a given acute angle in a right triangle, given the lengths of at least two of the sides?
- Find the values of the five other trigonometric ratios when given the value of one trigonometric ratio for an acute angle of a right triangle?

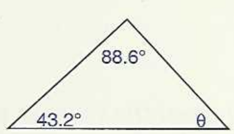
Exercise 5–1

Convert each angle to its measure in decimal degrees. Round the answer to the nearest 0.001° where necessary. Also, state whether each angle is acute or obtuse.

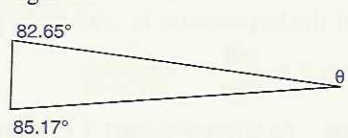
1. 13°25′	**2.** 111°56′	**3.** 0°12′	**4.** 42°37′
5. 25°33′19″	**6.** 87°2′13″	**7.** 165°47′	**8.** 19°15′
9. 33°5′55″	**10.** 0°19′12″	**11.** 159°59′	**12.** 20°1′

In the following problems find the measure of angle θ.

13.



14.



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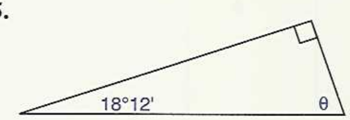

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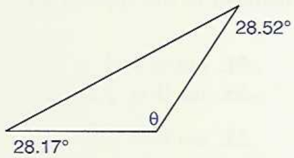
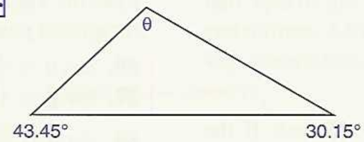
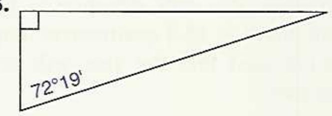


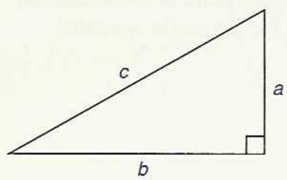
17.



18.



In the following problems find the length of the missing side. The lengths *a*, *b*, and *c* refer to the diagram. Leave your answer in exact form (in terms of rational numbers and radicals) unless the data is given in decimal form. In these cases, round the answer to the same number of decimal places as the data.

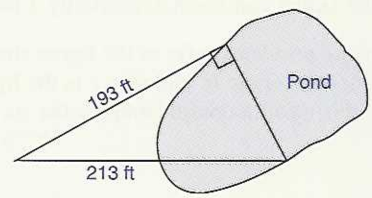


	a	*b*	*c*
19.	9	12	?
20.	10	?	26
21.	?	8	10
22.	5	10	?
23.	12	?	18
24.	?	6.8	9.2
25.	$\sqrt{5}$	3	?
26.	13.2	19.6	?
27.	100	150	?
28.	0.66	1.42	?
29.	$\sqrt{7}$	3	?
30.	4	?	$\sqrt{23}$
31.	6.3	?	15.0
32.	2	?	3
33.	$3\sqrt{2}$	$4\sqrt{5}$	?
34.	$3\sqrt{2}$	?	$4\sqrt{5}$
35.	?	19	28
36.	1,002	3,512	?
37.	1	1	?
38.	30	?	50

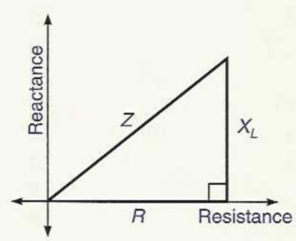
39. A flagpole is 55 feet tall and is supported by a wire attached to the top of the pole and to the ground 26 feet from the base of the pole. How long is the wire, to the nearest foot?

40. A flagpole is 93 feet tall and is supported by a wire that is 157 feet long, attached at the top of the pole and to the ground some distance from the base of the pole. Find the distance of the wire's ground attachment point from the base of the pole, to the nearest foot.

41. A surveyor has made the measurements shown in the diagram in order to compute the width of a pond. How wide is the pond, to the nearest 0.1 foot?



The diagram, called an impedance diagram, is used to compute total impedance Z in a certain electronic circuit. X_L is called the inductive reactance, and R is the resistance. All units are ohms. Use the diagram for problems 42–45.



42. If the inductive reactance X_L is 40.0 ohms and the resistance R is 56.6 ohms, calculate the total impedance Z to the nearest 0.1 ohm.

43. If $X_L = 5.68$ ohms and $R = 19.25$ ohms, find Z to the nearest 0.01 ohms.

44. If $Z = 213$ ohms and $R = 183$ ohms, find X_L to the nearest ohm.

45. If $Z = 4{,}340$ ohms and $X_L = 2{,}150$ ohms, find R to the nearest 10 ohms.

46. A machinist has to cut a rectangular piece of steel along its diagonal. The saw that will be used can cut this thickness of steel at the rate of 0.75 inch per minute. If the piece is 13.8 inches long and 9.6 inches wide, calculate how many minutes, to the nearest minute, it will take to cut the piece.

47. Assume the same situation as problem 46, except that the piece is 15.0 centimeters long and 10.5 centimeters wide, and that the saw will cut 0.8 centimeters per minute.

48. The ladder on a fire truck can extend 125 feet. If the truck is 25 feet from a building, how high up the building can the ladder reach, to the nearest tenth of a foot?

49. If the fire truck of problem 48 moves 5 more feet from the building (to 30 feet), does the height up the building that the ladder can reach decrease by 5 feet?

Find the value of the other member of the appropriate reciprocal pair.

50. $\cos \theta = \frac{1}{2}$

51. $\cos \alpha = \frac{1}{3}$

52. $\csc \beta = 1.5$

53. $\tan \beta = \sqrt{2}$

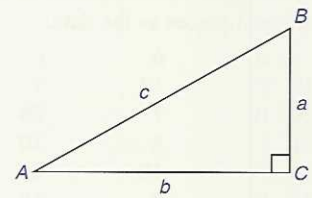
54. $\sin \beta = \dfrac{\sqrt{3}}{5}$

55. $\cot \theta = \sqrt{5}$

56. $\tan \alpha = 2.25$

57. $\cos \theta = \dfrac{\sqrt{6}}{8}$

The following problems refer to the figure shown. Observe that side a is opposite angle A, side b is opposite angle B, and side c is the hypotenuse. You are given parts of right triangle ABC. Use this information to compute the six trigonometric ratios for the angle specified.



	a	b	c	Find ratios for this angle
58.	3	4		A
60.	1	3		B
62.	5	$\sqrt{7}$		B
64.	2		$\sqrt{5}$	A
66.	12	13		A
68.		6	10	B
70.	$\sqrt{3}$	4		A
72.	1		2	B
74.	9	5		B
76.	x	y		A
78.		y	$2y$	B

	a	b	c	Find ratios for this angle
59.	5		13	B
61.	4	$\sqrt{10}$		B
63.		2	$\sqrt{13}$	B
65.	4		7	A
67.	5	12		B
69.	10		15	A
71.	1	1		B
73.		5	8	A
75.	2		8	B
77.	x		z	B
79.	$\frac{z}{3}$		z	A

In the following problems you are given one of the trigonometric ratios for an angle. Use this (a) to sketch a triangle for which the ratio is true, then (b) use this triangle to find the other five trigonometric ratios for that angle.

80. $\sin \alpha = \frac{4}{5}$ **81.** $\cos \alpha = \frac{1}{4}$ **82.** $\cos \alpha = 0.5$ **83.** $\tan \alpha = 4$ **84.** $\sec \alpha = 3$

85. $\cos \alpha = 0.2$ **86.** $\sin \alpha = \dfrac{5}{13}$ **87.** $\csc \alpha = 1.6$ **88.** $\cos \alpha = 0.9$ **89.** $\cos \alpha = \dfrac{\sqrt{3}}{5}$

90. In right triangle ABC, $\sin A = x$ and $x > 0$. Sketch a triangle for which this is true and use it to find $\sin B$ in terms of the variable x. $\left(\textit{Hint: } x = \dfrac{x}{1}.\right)$

91. In right triangle ABC, $\tan A = x$ and $x > 0$. Sketch a triangle for which this is true and use it to find $\sin B$ in terms of the variable x.

92. In right triangle ABC, $\sec A = x$ and $x > 0$. Sketch a triangle for which this is true and use it to find $\sec B$ in terms of the variable x.

93. In right triangle ABC, $\tan B = x$ and $x > 0$. Sketch a triangle for which this is true and use it to find $\tan A$ in terms of the variable x.

Solve the following problems.

94. Find the ground speed, to the nearest knot, of an aircraft flying with a heading due east and an airspeed of 132 knots, if there is a wind blowing from the north at 23 knots.

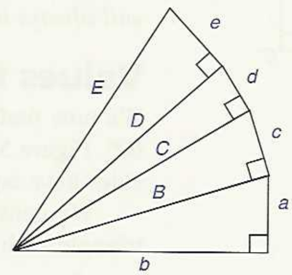
95. Find the ground speed, to the nearest knot, of an aircraft flying with a heading due west and an airspeed of 105 knots, if there is a wind blowing from the north at 18 knots.

96. Find the ground speed, to the nearest knot, of an aircraft flying with a heading due south and an airspeed of 178 knots, if there is a wind blowing from the east at 25 knots.

$\boxed{97.}$ Find the speed relative to the ground, to the nearest 0.1 knot, of a boat heading straight across a river at 16 knots if the current is moving at 4.3 knots.

98. To the nearest 0.1 inch, find the length of the diagonal of an $8\frac{1}{2}$ inch by 11 inch piece of paper.

99. In the "Mathematics of Warfare" by F. W. Lanchester (from *The World of Mathematics* by James R. Newman) Mr. Lanchester presents the idea that, all other things being equal, the strengths of fighting forces add in a manner proportional to the squares of their numbers. Referring to the diagram, this means that a force of size B is equal to two forces of sizes a and b; that C is equal to the combined strengths of a, b, and c, and so forth. Assuming that forces a, b, c, d, and e are of size 20, 5, 12, 8, and 10, respectively, find the size of force E that is equivalent to the combined strengths of these forces. Find this force to the nearest unit.

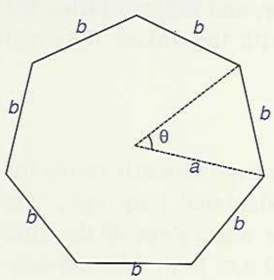


Skill and review

1. Find the equation of the line that passes through the points $(-2,5)$ and $(3,-10)$.
2. Solve the inequality $|2x - 3| < 13$.
3. Graph the parabola $y = x^2 - 3x - 4$.
4. Use the rational zero theorem and synthetic division to find all zeros of the polynomial $2x^4 + 5x^3 - 5x^2 - 5x + 3$.
5. Graph the polynomial function $f(x) = (x + 1)(x - 2)^2(x + 3)$.

5–2 Angle measure and the values of the trigonometric ratios

The diagram illustrates a piece of wood that is being mass produced to form the bottom of a planter. Find dimension a in the figure if $b = 8\frac{1}{4}$ inches.



The mathematics necessary to solve this problem involve using the trigonometric ratios. We will see how to solve this problem in this section, after studying some basic principles on which we build our theory.

Trigonometric ratios for equal angles are equal

It is possible to have angles of the same measure in different right triangles. Angles A and A' in figure 5–7 are examples. The trigonometric ratios will have the same value for these angles. This is because these triangles are *similar*—that is, they have the same shape, but perhaps different sizes. It is a theorem of geometry that corresponding ratios in similar figures are equal. Thus, in

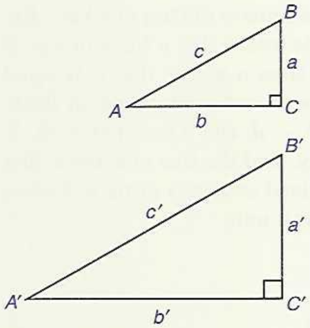


Figure 5–7

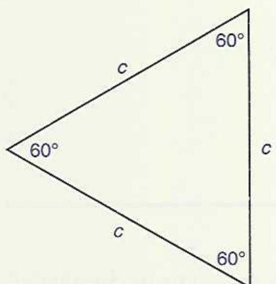


Figure 5–8

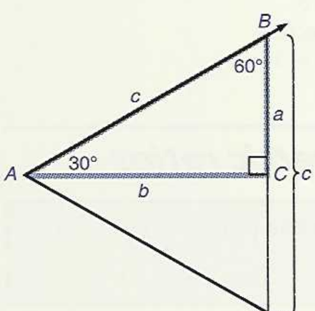


Figure 5–9

	Sine	Cosine	Tangent
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Table 5–1

figure 5–7, $\frac{a}{c} = \frac{a'}{c'}$, so $\sin A = \sin A'$. For the same reasons, the other five trigonometric ratios are also equal.

Note Read A' as "A-prime," B' as "B-prime," etc.

These facts mean that the values of the trigonometric ratios for an acute angle do not depend on the particular right triangle in which it appears. *For an acute angle with a given measure, the values of the trigonometric ratios will always be the same.*

Values for angles of measure 30°, 45°, and 60°

We now find the trigonometric ratios for some special angles—30°, 45°, and 60°. Figure 5–8 shows an equilateral triangle, which is a triangle in which all sides have equal length. We label this length c.

We construct the line AC, as shown in figure 5–9, which forms a right triangle with acute angles $A = 30°$ and $B = 60°$. The length a is half of c, so $a = \frac{c}{2}$. We find b next.

$$b^2 = c^2 - a^2 = c^2 - \left(\frac{c}{2}\right)^2, \text{ so } b^2 = \frac{3c^2}{4}, \text{ or } b = \frac{\sqrt{3}}{2}c.$$

Angle A is a 30° angle, so we will write sin 30° to mean the value of the sine ratio associated with an angle of measure 30°. Thus,

$$\sin 30° = \frac{a}{c} = \frac{\frac{c}{2}}{c} = \frac{1}{2}; \cos 30° = \frac{b}{c} = \frac{\frac{\sqrt{3}}{2}c}{c} = \frac{\sqrt{3}}{2}$$

$$\tan 30° = \frac{a}{b} = \frac{\frac{c}{2}}{\frac{\sqrt{3}}{2}c} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

The values for a 60° angle can be obtained from angle B in figure 5–9.

In the exercises we will compute the sine, cosine, and tangent ratios for a 45° angle; these are shown in table 5–1, along with the values obtained above.

General values

It is actually impossible to find the exact values of the trigonometric ratios for most angles. Tables of approximate values were calculated long ago. The earliest known table of trigonometric values, for the equivalent of the sine ratio, was created by Hipparchus of Nicaea about 150 B.C. In the second century A.D., Ptolemy constructed a table of values of the sine ratio for acute angles in increments of one-quarter degree. Today we use calculators to approximate these values. When using a calculator it is important that the calculator be in **degree mode.** This means that the calculator is expecting the

measure of the angle in decimal degrees. *Check your calculator's manual to make sure it is in degree mode.* This is usually done with a key marked $\boxed{\text{DRG}}$ or simply $\boxed{\text{DEG}}$. "DRG" means degrees, radians, grads. We discuss radian measure in a later section. Grads, or grades, is the metric measure for an angle. There are 100 grads in a right angle. We will not use this measure in this text.

To select degree mode on the Texas Instruments TI-81 it is necessary to select "Deg" under the $\boxed{\text{MODE}}$ feature. TO do this, select $\boxed{\text{MODE}}$, darken in the "Deg" mode indicator (use the four cursor moving "arrow" keys) and select $\boxed{\text{ENTER}}$.

Example 5–2 A shows typical calculator keystrokes to calculate each value.

■ ***Example 5–2 A***

Find each value rounded to four decimal places.

1. sin 34.51° 34.51 $\boxed{\sin}$ Display $\boxed{0.5665500655}$

sin 34.51° ≈ 0.5666 TI-81: $\boxed{\text{SIN}}$ 34.51 $\boxed{\text{ENTER}}$

2. sec 33.5°

Since there is no secant key on a calculator we use the fact that sec 33.5° $= \dfrac{1}{\cos 33.5°}$. Compute cos 33.5° and divide it into one; the $\boxed{1/x}$ key is designed for this type of situation.

33.5 $\boxed{\cos}$ $\boxed{1/x}$ Display $\boxed{1.199204943}$

TI-81: $\boxed{(}$ $\boxed{\text{COS}}$ 33.5 $\boxed{)}$ $\boxed{x^{-1}}$

Note that on the TI-81 the $\boxed{1/x}$ key is the $\boxed{x^{-1}}$ key.

sec 33.5° ≈ 1.1992

3. cos 13°43′

Recall that angles in the DMS system must be converted to decimal degrees. We show the calculation with and without special calculator keys. (See also example 5–1 A.)

No special keys: 13 $\boxed{+}$ 43 $\boxed{\div}$ 60 $\boxed{=}$ $\boxed{\cos}$

Display $\boxed{0.9714801855}$

Calculator A: 13 $\boxed{\circ\ '\ ''}$ 43 $\boxed{\circ\ '\ ''}$ $\boxed{\cos}$

Calculator B: 13.43 $\boxed{\to\text{H}}$ $\boxed{\cos}$

TI-81: $\boxed{\text{COS}}$ $\boxed{(}$ 13 $\boxed{+}$ 43 $\boxed{\div}$ 60 $\boxed{)}$ $\boxed{\text{ENTER}}$

cos 13°43′ ≈ 0.9715 ■

Finding an angle from a known trigonometric ratio

It is important to be able to reverse the operations discussed above. For example, if θ is an acute angle and sin θ = $\frac{1}{2}$, what is θ? We can see from table 5–1 that θ must be 30°. The calculator is programmed to solve this problem.

This is done with the *inverse trigonometric ratios* called the inverse sine ($\sin^{-1}$), inverse cosine ($\cos^{-1}$), and inverse tangent ($\tan^{-1}$) ratios. The superscript -1 does *not* indicate a reciprocal value in the way that, say, $2^{-1} = \frac{1}{2}$. We will study these ratios in more detail later. For now we illustrate how to find the acute angle whose sine, cosine, or tangent value is known.

$\sin^{-1}$	$\cos^{-1}$	$\tan^{-1}$
$\boxed{\sin}$	$\boxed{\cos}$	$\boxed{\tan}$

For this, most calculators use the appropriate key (sin, cos, tan), prefixed by another key such as $\boxed{\text{SHIFT}}$, $\boxed{\text{2nd}}$, $\boxed{\text{INV}}$, or $\boxed{\text{ARC}}$. The appropriate function is generally shown above the key itself. The *result is always an angle in decimal degrees* (when the calculator is in degree mode). We will show the necessary two keystrokes as one.

■ Example 5–2 B

Find θ in the following problems using the calculator. Assume θ is an acute angle. Round the answer to the nearest 0.01°.

1. $\cos\theta = 0.4602$

We need to calculate $\cos^{-1}0.4602$.

0.4602 $\boxed{\cos^{-1}}$ Display $\boxed{62.59998611}$

TI-81: $\boxed{\text{COS}^{-1}}$ .4602 $\boxed{\text{ENTER}}$

$\theta \approx 62.60°$

2. $\csc\theta = 1.0551$

We use the fact that if $\csc\theta = 1.0551$, then $\sin\theta = \frac{1}{1.0551}$. Thus we compute $\sin^{-1}(\frac{1}{1.0551})$. Use the $\boxed{1/x}$ key.

1.0551 $\boxed{1/x}$ $\boxed{\sin^{-1}}$ Display $\boxed{71.40162609}$

TI-81: $\boxed{\text{SIN}^{-1}}$ 1.0551 $\boxed{x^{-1}}$ $\boxed{\text{ENTER}}$

$\theta \approx 71.40°$ ■

Solving right triangles

One application of trigonometry that occurs in many situations is *solving right triangles*—this means *discovering the lengths of all sides and the measures of all angles* of the triangle. We will round the values we compute to the same number of decimal places as the given data.

These types of situations fall into two categories, ones in which we know one side and one acute angle and others in which we know two sides and no angles. Each category is illustrated in example 5–2 C.

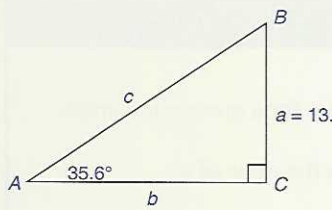
Note In general we assume side *a* is opposite angle *A*, side *b* is opposite angle *B*, and the hypotenuse *c* is opposite right angle *C*.

■ Example 5–2 C

Solve the following right triangles.

1. $A = 35.6°$, $a = 13.6$ (one side, one angle)

To solve this triangle we need to find the lengths of sides b and c and the measure of angle B. Since angle C is always 90°, angles A and B total 90°. Thus, angle B is $90° - 35.6° = 54.4°$.



We now note that $\sin A = \dfrac{a}{c}$, so that

$$\sin 35.6^\circ = \frac{13.6}{c}$$

$$c \sin 35.6^\circ = 13.6 \quad \text{Multiply each member by } c$$

$$c = \frac{13.6}{\sin 35.6^\circ} \quad \text{Divide each member by } \sin 35.6^\circ$$

$$c \approx 23.4$$

13.6 $\boxed{\div}$ 35.6 $\boxed{\text{sin}}$ $\boxed{=}$ Display $\boxed{23.3627613}$

TI-81: 13.6 $\boxed{\div}$ $\boxed{\text{SIN}}$ 35.6 $\boxed{\text{ENTER}}$

Now we find b by noting that $\tan A = \dfrac{a}{b}$.

$$\tan 35.6^\circ = \frac{13.6}{b}$$

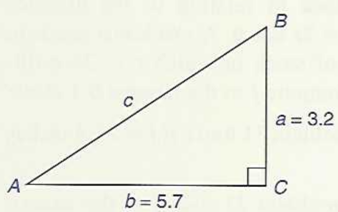
$$b \tan 35.6^\circ = 13.6$$

$$b = \frac{13.6}{\tan 35.6^\circ}$$

$$b \approx 19.0$$

Since we know the lengths of all sides and all angles we have solved the triangle. To summarize, $a = 13.6$, $b \approx 19.0$, $c \approx 23.4$, $A = 35.6^\circ$, $B = 54.4^\circ$, $C = 90^\circ$.

2. $a = 3.2$, $b = 5.7$ (two sides)



We can find the length of side c by the Pythagorean theorem: $c \approx 6.5$.

We can find angle A by noting that $\tan A = \dfrac{a}{b}$.

$$\tan A = \tfrac{3.2}{5.7}$$

$$A = \tan^{-1}\left(\tfrac{3.2}{5.7}\right)$$

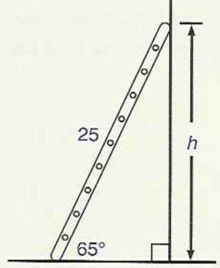
3.2 $\boxed{\div}$ 5.7 $\boxed{=}$ $\boxed{\tan^{-1}}$ Display $\boxed{29.31000707}$

TI-81: $\boxed{\text{TAN}^{-1}}$ $\boxed{(}$ 3.2 $\boxed{\div}$ 5.7 $\boxed{)}$ $\boxed{\text{ENTER}}$

$$A \approx 29.3^\circ$$

$B = 90^\circ - 29.3^\circ = 60.7^\circ$. Thus, $a = 3.2$, $b = 5.7$, $c \approx 6.5$, $A \approx 29.3^\circ$, $B \approx 60.7^\circ$, $C = 90^\circ$.

3. A tag on a 25-foot ladder states that, for safety reasons, the angle that the ladder makes with the ground should not exceed 65°. How high can the ladder reach without exceeding this angle, to the nearest 0.1 feet?



We need to find h in the figure. If we observe that h is opposite the known angle and that the length of the hypotenuse of the triangle is known, we see that we can use the sine ratio.

$$\sin 65^\circ = \frac{h}{25}$$

$$25 \sin 65^\circ = h \quad \text{Multiply each member by 25}$$

$$22.7 \approx h$$

The ladder can reach a height of approximately 22.7 feet without exceeding a 65° angle with the ground. ■

Mastery points

Can you

- Compute the value of the trigonometric ratios for a given acute angle, using a calculator?
- Compute the value of an acute angle, given the value of a trigonometric ratio, using a calculator?
- Solve a right triangle when given one side and one acute angle?
- Solve a right triangle when given two sides?

Exercise 5–2

Use a calculator to find four-decimal-place approximations for the following.

1. sin 31.28°	**2.** cos 85.23°	**3.** tan 11.95°	**4.** sec 40.08°
5. cot 28.87°	$\boxed{6.}$ csc 5.15°	**7.** sin 40.28°	**8.** tan 76.23°
9. sec 66.47°	**10.** sin 35.56°	**11.** sin 78.33°	**12.** cos 17.45°
13. sin 35°56′	**14.** sin 78°33′	**15.** cos 17°45′	**16.** cos 85°28′
17. tan 40°41′	**18.** tan 35°8′	**19.** cos 23°24′	**20.** cos 56°24′
$\boxed{21.}$ cot 13°3′	**22.** sin 48°8′	**23.** tan 33°38′	**24.** sec 86°22′

25. A surveyor needs to compute R in the following formula as part of finding the area of the segment of a circle: $R = \dfrac{LC}{2 \sin I}$. Find R to one decimal place if $LC = 425.0$ feet and $I = 13.2°$.

26. Compute R using the formula of the previous problem if $LC = 611.1$ meters and $I = 18°20'$. Round the answer to two decimal places.

27. In the mathematical modeling of an aerodynamics problem the following equation arises:

$$y = x \cos A \cos B - x^2 \cos A \sin B - x^3 \sin A.$$

Compute y to two decimal places if $x = 2.5$, $A = 31°$, and $B = 17°$.

28. Compute y to two decimal places using the formula of problem 27 if $x = 1.2$, $A = 10°$, and $B = 15°$.

29. The average power in an AC circuit is given by the formula $P = VI \cos \theta$. Compute P (in watts) if $V = 120$ volts, $I = 2.3$ amperes, and $\theta = 45°$, to the nearest 0.1 watt.

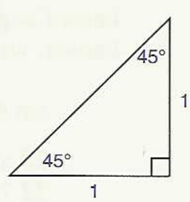
30. Compute P using the formula of the previous problem if $V = 42$ volts, $I = 25$ amperes, and $\theta = 45°$, to the nearest 0.1 watt.

31. A formula that relates the distance across the flats of a piece of hexagonal stock in relation to the distance across the corners is $f = 2r \cos \theta$. A machinist needs to compute f for a piece of stock in which $r = 28$ millimeters and $\theta = 30°$. Compute f to the nearest 0.1 mm.

32. Using the formula of problem 31 find r if $f = 21.4$ inches and $\theta = 25°$.

33. Using the formula of problem 31 find θ to the nearest 0.1 if $f = 36.8$ millimeters and $r = 24.0$.

34. Find the exact values for the sine, cosine, and tangent ratios for an angle of measure 45° by proceeding in the following manner. Draw an isosceles right triangle—a right triangle in which the two legs have the same length. Label this length one. Observe that the two acute angles must be 45°. Now find the length of the hypotenuse and use the definitions of the trigonometric ratios to find the desired values.



Find the unknown acute angle θ to the nearest 0.01°.

35. $\sin\theta = 0.8007$ **36.** $\cos\theta = 0.1028$ **37.** $\tan\theta = 1.8807$ **38.** $\sin\theta = 0.9484$

39. $\cos\theta = 0.8515$ **40.** $\tan\theta = 1.0014$ **41.** $\sin\theta = \frac{35.9}{68.3}$ $\boxed{\textbf{42.}}$ $\cos\theta = \frac{8.25}{12.5}$

43. $\tan\theta = 2$ $\boxed{\textbf{44.}}$ $\csc\theta = 1.1243$ **45.** $\sec\theta = 4.8097$ **46.** $\cot\theta = 2.5$

47. $\sec\theta = \frac{6.45}{2.35}$ **48.** $\csc\theta = \sqrt{10.8}$

In the following problems you are given one side and one angle of a right triangle. Solve the triangle. Round all answers to the same number of decimal places as the data.

$\boxed{\textbf{49.}}$ $a = 15.2, B = 38.3°$ **50.** $a = 12.6, B = 17.9°$ **51.** $a = 11.1, A = 13.7°$ **52.** $a = 5.25, A = 70.3°$

53. $b = 0.672, A = 29.4°$ **54.** $b = 15.2, A = 81.3°$ **55.** $b = 21.8, B = 78.0°$ **56.** $b = 2.14, B = 50.4°$

57. $c = 10.0, A = 15.0°$ **58.** $c = 3.45, A = 46.2°$ $\boxed{\textbf{59.}}$ $c = 122, B = 65.5°$ **60.** $c = 31.5, B = 62.0°$

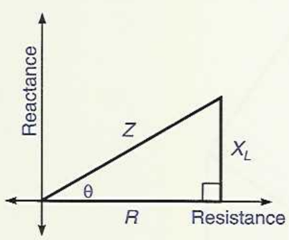
In the following problems you are given two sides of a right triangle. Solve the triangle. Round all lengths to the same number of decimal places as the data and all angles to the nearest 0.1°.

61. $a = 13.1, b = 15.6$ **62.** $a = 5.67, b = 8.91$ **63.** $a = 0.22, b = 1.34$ **64.** $a = 2.82, b = 1.09$

65. $a = 17.8, c = 25.2$ **66.** $a = 311, c = 561$ $\boxed{\textbf{67.}}$ $b = 51.3, c = 111.0$ **68.** $b = 4.55, c = 5.66$

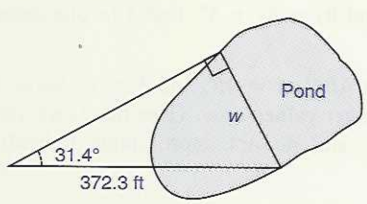
69. $a = 12.0, c = 13.0$ **70.** $a = 33.1, c = 41.0$ **71.** $b = 84.0, c = 90.1$ **72.** $b = 0.651, c = 0.927$

73. The figure illustrates an impedance diagram used in electronics theory. If Z (impedance) = 10.35 ohms and X_L (inductive reactance) = 4.24 ohms, find θ (phase angle) to the nearest degree and R (resistance) to the nearest 0.01 ohm.

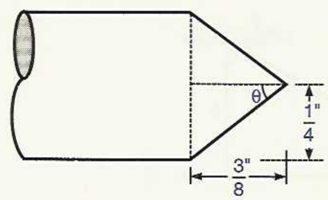


74. Use the impedance diagram of problem 73 to find Z if $\theta = 24.2°$ and $X_L = 22.6$ ohms.

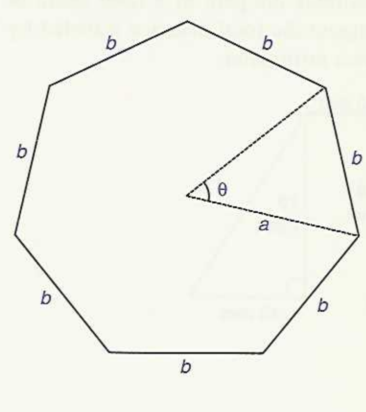
75. The diagram illustrates the measurements a surveyor made to find the width w of a pond; compute the width to the nearest foot.



76. The diagram illustrates the tip of a threading tool; find angle θ to the nearest degree.

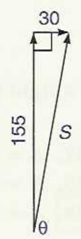


77. The diagram illustrates a piece of wood that is being mass produced to form the bottom of a planter. Find dimension a in the figure to the nearest 0.01 inch if $b = 8\frac{1}{4}$ inches. Note that angle θ is $\frac{360°}{7}$.

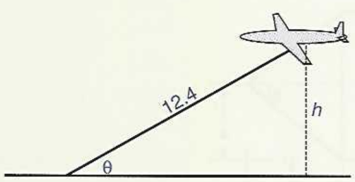


78. A formula found in electronics is $E = \dfrac{P}{I\cos\theta}$, where E is voltage, P is power, I is current, and θ is phase angle. Find E (in volts) if $P = 45.0$ watts, $I = 2.5$ amperes, and $\theta = 15°$. Round the answer to the nearest 0.1 volt.

79. The diagram illustrates the wind triangle problem in air navigation. A plane has an airspeed of 155 mph and heading of due north. It is flying in a wind from the west with a speed of 30 mph. Find the ground speed S and the ground direction θ, each to the nearest unit.

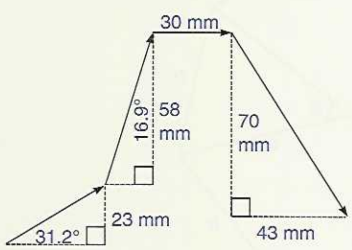


80. An *angle of elevation* is an angle formed by one horizontal ray and another ray that is above the horizontal. Angle θ in the diagram is the angle of elevation to an aircraft that radar shows has a *slant distance* of 12.4 miles from the radar site. If θ is 30.1°, find the elevation h of the aircraft, to the nearest 100 feet. Remember that 1 mile = 5,280 feet.

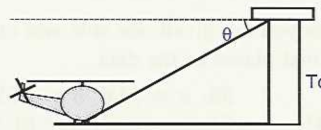


81. If it is known that an aircraft is flying at 28,500 feet and the angle of elevation of a radar beam tracking the aircraft is 8.2°, what is the slant distance d from the radar to the aircraft, to the nearest 100 feet? See problem 80.

82. The diagram illustrates the path of a laser beam on an optics table. Compute the total distance traveled by the beam to the nearest millimeter.

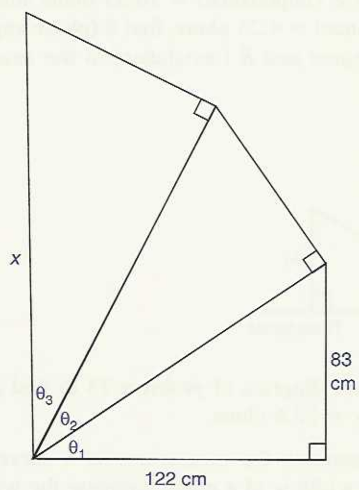


83. An *angle of depression* is an angle formed by one horizontal ray and another ray that is below the horizontal. Angle θ in the figure is the angle of depression formed by the line of sight of an observer in an airport control tower looking at a helicopter on the ground. If θ is 17.2° and the tower is 257 feet high, how far is the aircraft from the base of the tower, to the nearest foot?



84. If an aircraft is 1.23 miles from the foot of the tower in problem 83, what is θ, to the nearest 0.1°? (Remember, 1 mile = 5,280 ft.)

85. The diagram is a top view of a portion of a spiral staircase that an architect has designed. If $\theta_1 = \theta_2 = \theta_3$, find the length x, to one decimal place. (Caution: Carry out your calculations to as many digits as practical to avoid an accumulation of errors.)



86. If the architect of problem 85 revises the plans so that $\theta_2 = \theta_1 + 5°$ and $\theta_3 = \theta_2 + 5°$, find x to one decimal place.

87. In right triangle ABC, $A = 45°$ and $b = 4$. Solve this triangle using *exact* values only. (Use the exact values for sin 45° etc. and do not approximate radicals as decimals.)

88. In right triangle ABC, $B = 60°$ and $b = 8$. Solve this triangle using *exact* values only. (Use the exact values for sin 60° etc. and do not approximate radicals as decimals.)

Skill and review

1. Graph the rational function $f(x) = \dfrac{2}{x^2 - 9}$.
2. Find the slope of the straight line $3x - 2y = 5$.
3. Solve the nonlinear inequality $x^2 - 2x > 3$.
4. In right triangle ABC, $a = 9.0$ and $b = 16.8$. Find c.

5–3 The trigonometric functions—definitions

In an electronic circuit with an inductive component to the impedance, the current follows the voltage. The difference is often measured in degrees. For example, the current may follow the voltage by 15°. In this case, we could say the phase angle of the current is −15°, relative to the voltage. We could just as easily say that the phase angle of the voltage is 345°, relative to the current. Find the phase angle of the voltage relative to the current if the phase angle of the current relative to the voltage is −88°.

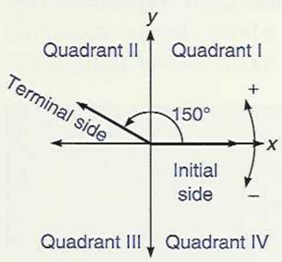


Figure 5–10

The relationship between current and voltage is one phenomenon that can be described with some of the terminology we study in this section.

Angles in standard position

As in the example above, there are many situations where we have to think of angles as being nonacute. This is often a situation in which we wish to describe an amount of rotation. For example, a ship may turn through an angle of 215°, a computed tomography (CT) scanner used in medical diagnosis may move through an angle of 360°, or a surveyor may find the measure of the angle at one corner of a piece of land to be 165°20′. For these situations we often place the angle in a rectangular (x-y) coordinate system.

Angle in standard position

An angle in standard position is formed by two rays, with the vertex at the origin. One ray always lies on the nonnegative portion of the x-axis. This ray is called the **initial side** of the angle. The second ray is called the **terminal side** of the angle. It may be in any quadrant or along any axis.

Figure 5–10 shows an angle in standard position, with measure 150°. We generally use the word "angle" instead of the phrase "angle in standard position."

In some situations it is convenient to let the measure of an angle be negative. If the measure of the angle is positive, we picture the terminal side as having moved away from the initial side in a counterclockwise direction (↶); if the measure of the angle is negative we picture the terminal side as having moved away from the initial side in a clockwise direction (↷). If an angle's measure is greater than 360° or less than −360° we consider the angle to have "gone around" more than once. Several examples of angles in standard position are shown in figure 5–11. In part c of the figure we show the angle as a 360° revolution, followed by an additional 200° turn.

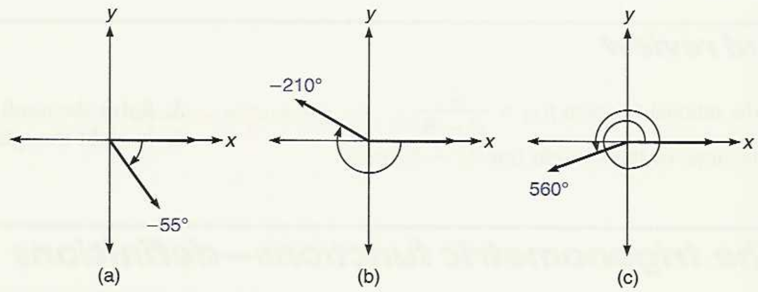


Figure 5–11

Angles that have the same terminal side are said to be **coterminal.** (All angles in standard position have the same initial side.) The 150° angle in figure 5–10 and the −210° angle in figure 5–11(b) are coterminal. We can see this when we realize that in each case the angle formed by the negative side of the x-axis and the terminal side of each angle is 30°. Since ±360° represents one complete revolution, coterminal angles are angles whose degree measures differ by an integer multiple of 360°. This forms the basis for our definition.

> **Coterminal angles**
>
> Two angles α and β are said to be coterminal if
>
> $$\alpha = \beta + n(360°), \quad n \text{ an integer}$$
>
> ***Concept***
>
> Two angles are coterminal if the measure of one can be formed from the measure of the other by adding or subtracting multiples of 360°.

■ ***Example 5–3 A***

In each case find a coterminal angle with measure x such that $0° \leq x < 360°$.

1. 875°

$875° - 360° = 515°$ Subtract 360° until x is found

$515° - 360° = 155°$ The required angle is 155°

We could have done this more elegantly by computing $875° - 2(360°)$.

2. −1,000°

$-1{,}000° + 360° = -640°$ Add 360° until x is found

$-640° + 360° = -280°$

$-280° + 360° = 80°$

Or compute as $-1{,}000° + 3(360°) = 80°$. ■

The trigonometric functions

We now define the six trigonometric functions. They have the same names as the six trigonometric ratios, and the same abbreviations.

The trigonometric ratios are actually functions with domain the set of *acute* angles. The trigonometric functions have the set of *all* angles as their domain. For acute angles the trigonometric functions are the same as the trigonometric ratios. The following definition refers to figure 5–12.

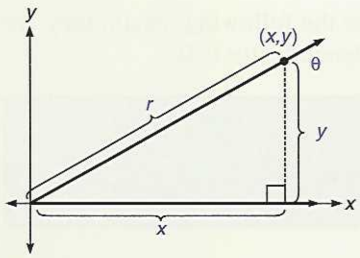


Figure 5–12

The trigonometric functions

Let θ be an angle in standard position, and let (x,y) be any point on the terminal side of the angle, except (0,0). Let $r = \sqrt{x^2 + y^2}$ be the distance from the origin to the point. Then,

$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}, \quad \tan\theta = \frac{y}{x}$$

$$\csc\theta = \frac{r}{y}, \quad \sec\theta = \frac{r}{x}, \quad \cot\theta = \frac{x}{y}$$

Note
1. We define r so that $r > 0$.
2. If x or y in the point (x,y) is zero, then those ratios with x or y in the denominator are not defined.
3. Unlike the trigonometric ratios, the trigonometric functions can take on negative values.

It can be proven that for a given angle, *it does not matter what point on the terminal side is chosen; the values of the trigonometric functions will be the same.* This is for the same reasons that the trigonometric ratios do not depend on the size of the right triangle.

It can also be seen that *coterminal angles have the same values for the trigonometric functions.* This is because two coterminal angles have the same terminal side, and the definitions depend solely upon a point on the terminal side.

The definitions of the trigonometric functions imply the following identities for all values of θ for which any denominator is nonzero. These identities look identical to those for the trigonometric ratios. They can be applied to either trigonometric ratios or functions.

Reciprocal function identities

$$\csc\theta = \frac{1}{\sin\theta}, \quad \sin\theta = \frac{1}{\csc\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}, \quad \cos\theta = \frac{1}{\sec\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}, \quad \tan\theta = \frac{1}{\cot\theta}$$

To see that the first reciprocal function identity is true for the trigonometric functions, observe that $\csc\theta = \frac{r}{y} = \frac{1}{\frac{y}{r}} = \frac{1}{\sin\theta}$. It is left as an exercise to show that the rest of these are true. Using the reciprocal function identities, we can usually find the values of the cosecant, secant, and cotangent functions by finding the reciprocal of the sine, cosine, and tangent functions. This will not work when a function has value zero, since the reciprocal of zero is not defined.

Two other identities that can be useful are the following; again, they are true only for those values of θ for which no denominator is 0.

Tangent/cotangent identities

$$\tan \theta = \frac{\sin \theta}{\cos \theta}, \quad \cot \theta = \frac{\cos \theta}{\sin \theta}$$

■ *Example 5–3 B*

Show that $\tan \theta = \dfrac{\sin \theta}{\cos \theta}$ is an identity for the trigonometric functions.

We show that each side of the equation is equivalent to the same thing.

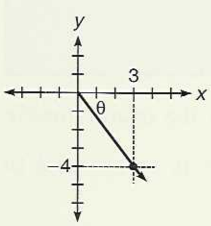
$$\begin{array}{ccl} \tan \theta & \dfrac{\sin \theta}{\cos \theta} & \\ \dfrac{y}{x} & \dfrac{\frac{y}{r}}{\frac{x}{r}} & \tan \theta = \frac{y}{x},\ \sin \theta = \frac{y}{r},\ \cos \theta = \frac{x}{r} \\ & \dfrac{y}{r} \cdot \dfrac{r}{x} & \text{Algebra of division} \\ & \dfrac{y}{x} & \text{Reduce } \frac{ry}{rx} \end{array}$$

Thus $\tan \theta = \dfrac{y}{x}$ and $\dfrac{\sin \theta}{\cos \theta} = \dfrac{y}{x}$, so $\tan \theta = \dfrac{\sin \theta}{\cos \theta}$. ■

Example 5–3 C illustrates finding values of the trigonometric functions for an angle in standard position, given a point on the terminal side of the angle.

■ *Example 5–3 C*

In each problem a point on the terminal side of an angle θ is given. Use it to find the trigonometric functions for that angle. Also, make a sketch of the angle.



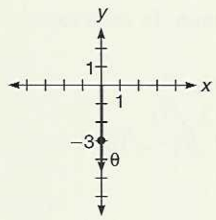
1. (3, −4)

$$\begin{aligned} r &= \sqrt{x^2 + y^2} && \text{Definition} \\ &= \sqrt{3^2 + (-4)^2} && \text{Replace } x, y \\ &= 5 \end{aligned}$$

$$\sin \theta = \frac{y}{r} = -\frac{4}{5}, \ \csc \theta = \frac{1}{\sin \theta} = -\frac{5}{4}$$

$$\cos \theta = \frac{x}{r} = \frac{3}{5}, \ \sec \theta = \frac{1}{\sin \theta} = \frac{5}{3}$$

$$\tan \theta = \frac{y}{x} = -\frac{4}{3}, \ \cot \theta = \frac{1}{\tan \theta} = -\frac{3}{4}$$



2. (0, −3)

$$r = \sqrt{0^2 + (-3)^2} = 3$$

$$\sin\theta = \frac{y}{r} = \frac{-3}{3} = -1,\ \csc\theta = \frac{1}{\sin\theta} = -1$$

$$\cos\theta = \frac{x}{r} = \frac{0}{3} = 0,\ \sec\theta = \frac{1}{\cos\theta} = \frac{1}{0}\text{; undefined}$$

$$\tan\theta = \frac{y}{x} = \frac{-3}{0}\text{, undefined; } \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{0}{-1} = 0$$

Mastery points

Can you

- When given an angle θ, find a positive coterminal angle with measure x such that $0° \le x < 360°$?
- Sketch an angle and find the values of the six trigonometric functions when given a point on the terminal side of the angle?

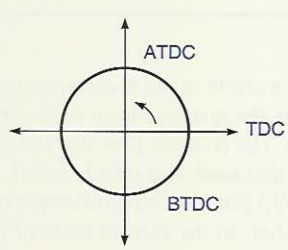
Exercise 5–3

In problems 1–17,

a. draw the initial and terminal side of the given angle.
b. state the measure of the smallest nonnegative angle that is coterminal with the given angle.

1. 420° **2.** −40° **3.** 230° **4.** 1,000° **5.** 800.6° **6.** 1,260°
7. 547.9° **8.** 2,000° **9.** −870° **10.** 625° **11.** 525° **12.** −610°
13. −530.3° **14.** 390° **15.** −720° **16.** −313° **17.** −11.9°

18. An automobile engine is timed to fire the spark plug for cylinder 1 at 8° BTDC (before top dead center), which, for our purposes, is −8°. Assuming this engine rotates in a counterclockwise direction, what is the equivalent amount ATDC (after TDC) (i.e., the least nonnegative angle coterminal with it)?



19. If an automobile engine is timed to fire at 13° BTDC, what is the equivalent amount ATDC?

20. If an automobile engine is timed to fire at 8.6° BTDC, what is the equivalent amount ATDC?

21. If an automobile engine is timed to fire at 6.1° BTDC, what is the equivalent amount ATDC?

22. In an electronic circuit with an inductive component to the impedance, the current follows the voltage. For example, the current may follow the voltage by 15°, in which case we could say the phase angle of the current is −15°, relative to the voltage. We could just as easily say that the phase angle of the voltage is 345°, relative to the current. Find the phase angle of the voltage relative to the current if the phase angle of the current relative to the voltage is (a) −88°, (b) −24.33°, (c) −35°56′, (d) −16.56°, (e) −0°14′, and (f) −0.14°. (Find the least nonnegative coterminal angle in each case.)

In the following problems you are given a point that lies on the terminal side of an angle in standard position. In each case, compute the value of all six trigonometric functions for the angle.

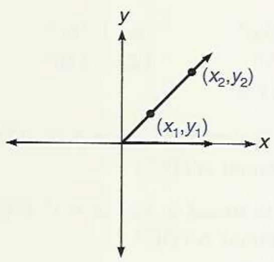
23. $(3,6)$ **24.** $(-2,5)$ **25.** $(-5,8)$ **26.** $(-7,-8)$ **27.** $(2,-2)$
28. $(3,0)$ **29.** $(-1,4)$ **30.** $(0,-4)$ **31.** $(-10,-15)$ **32.** $(3,\sqrt{5})$
33. $(-\sqrt{2},6)$ **34.** $(3,-\sqrt{6})$ **35.** $(-\sqrt{3},-\sqrt{2})$ **36.** $(1,-\sqrt{3})$ **37.** $(\sqrt{6},-\sqrt{10})$

Show that each identity is true.

38. $\sec\theta = \dfrac{1}{\cos\theta}$ **39.** $\cot\theta = \dfrac{1}{\tan\theta}$ **40.** $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ **41.** $\cos\theta = \dfrac{1}{\sec\theta}$ **42.** $\sin\theta = \dfrac{1}{\csc\theta}$

To solve the following two problems we must recall that the equation of a nonvertical straight line can be put in the form $y = mx + b$, where m is the slope and b is the y-intercept. If a straight line passes through the origin then $b = 0$ and the equation becomes $y = mx$.

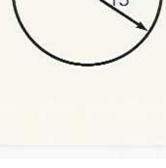
43. Show that if two different points lie on the terminal side of an angle in standard position, then using either point gives the same value for the sine function. For the sake of simplicity assume the terminal side is not vertical or horizontal. Represent the points as (x_1,y_1) and (x_2,y_2). Note that these points lie on the same line. The equation of any line that passes through the origin is of the form $y = mx$, so we know that for the same value of m, $y_1 = mx_1$ and $y_2 = mx_2$. This means that the points (x_1,y_1) and (x_2,y_2) can be rewritten as (x_1,mx_1) and (x_2,mx_2). Use these versions of the points to compute the length r for each point. Then show that the value of the sine function is the same when computed using either point.



44. Show that if the trigonometric function values are the same for two points, then these points lie on the terminal side of the same angle. Assume for simplicity that these points are not located on either the x-axis or the y-axis. To show that the points lie on the terminal side of the same angle we must show that both points lie on the same line and are in the same quadrant. Let (x_1,y_1) and (x_2,y_2) represent the two points, and consider the value of the tangent function as given by each point. This can be used to show that $y_1 = mx_1$ and $y_2 = mx_2$ (for the same value of m). This means that the two points lie on the same line. Now explain why they must be in the same quadrant.

Skill and review

1. In right triangle ABC, $a = 9.0$, $b = 16.8$. Solve the triangle.
2. Use the graph of $y = \sqrt{x}$ to graph the function $f(x) = \sqrt{x-4} - 2$.
3. Factor $8x^3 - 27$.
4. Rationalize the denominator of $\dfrac{3}{\sqrt{6}}$.
5. Compute $\dfrac{5-3^2}{8} - 2$.
6. The circumference C of a circle is the distance around the circle. The radius r is the distance from the center to the edge of the circle. The relation $C = 2\pi r$ has been known for several thousand years. (π is a real number, and $\pi \approx 3.14159$.) Find the circumference of a circle with radius 15 inches, to the nearest tenth of an inch.



5–4 Values for any angle—the reference angle/ASTC procedure

If a force of 200 pounds is applied to a rope to drag an object, the actual force tending to move the object horizontally is $f(\theta) = 200 \cos \theta$, where θ is the angle the rope makes with the horizontal. Compute the force tending to move the object horizontally if the angle of the rope is 25°.

As in this problem, many physical phenomena can be described using the trigonometric functions. In this section we study more about these functions.

The values of the trigonometric functions for an angle of any measure are related to the values for the acute angles of the first quadrant. These values (for the first quadrant) are the same as those for the trigonometric ratios for acute angles. The values of the trigonometric functions for any angle have a sign and a "size" (absolute value). We first discuss the sign of the basic trigonometric functions, then the size.

The ASTC rule—the signs of the trigonometric functions by quadrant

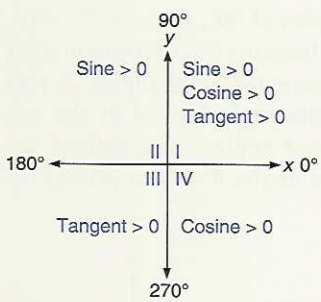


Figure 5–13

The **sign** of the value of a trigonometric function for an angle *depends on the quadrant in which the angle terminates.* Figure 5–13 shows the quadrants in which the sine, cosine, and tangent functions are positive. (They are negative in the other quadrants.) The figure shows that the sine function is positive in quadrants I and II, and therefore negative in quadrants III and IV. This is because the sine function is defined by the ratio $\frac{y}{r}$; since r is always positive this ratio is positive where y is positive, in quadrants I and II. Since the cosine function is $\frac{x}{r}$ and $r > 0$, the cosine is positive where x is positive: quadrants I and IV. The tangent function is defined by $\frac{y}{x}$, so it is positive where x and y are both positive (quadrant I) or both negative (quadrant III).

Figure 5–13 should be memorized. It is reflected in the following four statements, which we call the ASTC rule.

The ASTC rule

In quadrant I,	**A**ll the trigonometric functions are positive.
In quadrant II, the	**S**ine function is positive.
In quadrant III, the	**T**angent function is positive.
In quadrant IV the	**C**osine function is positive.

One memory aid is the sentence "**A**ll **S**tudents **T**ake **C**alculus."

Since the sign of the reciprocal of a value is the same as the value, the sign of the cosecant function is the same as the sign of the sine function, that of the secant function is the same as that of the cosine function, and the sign of the cotangent function is the same as that of the tangent function.

The ASTC rule can be used to determine in which quadrant a given angle terminates.

■ ***Example 5–4 A***

Determine in which quadrant the given angle θ terminates.

1. $\sin \theta < 0$, $\tan \theta > 0$

If $\sin \theta < 0$ then θ terminates in quadrants III or IV.
If $\tan \theta > 0$ then θ terminates in quadrants I or III.

Thus, for both conditions to be true, θ must terminate in quadrant III.

2. $\cos \theta < 0$, $\sin \theta > 0$

$\cos \theta < 0$ means θ terminates in quadrant II or III.
$\sin \theta > 0$ means θ terminates in quadrant I or II.

Thus, θ terminates in quadrant II. ■

Reference angles

Angles whose degree measures are integer multiples of 90°, such as 0°, ±90°, ±180°, ±270°, etc. are called **quadrantal angles** because their terminal sides fall between two quadrants. All other angles are nonquadrantal angles. A **reference angle** for a nonquadrantal angle is the acute angle formed by the terminal side of the angle and the x-axis. A reference angle is not defined for quadrantal angles. Figure 5–14 shows a reference angle, θ′ (theta-prime) for an angle θ terminating in each quadrant.

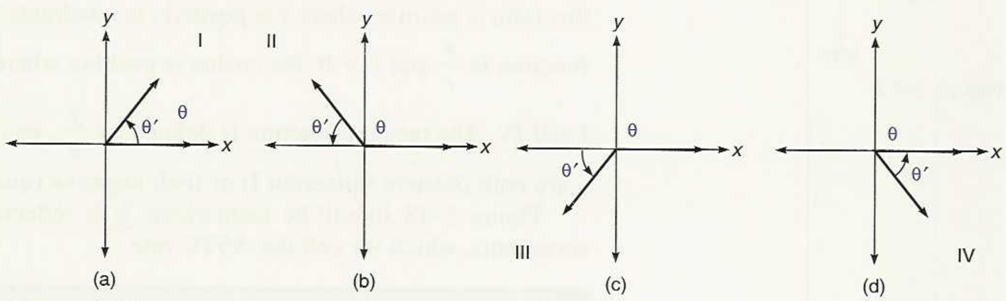
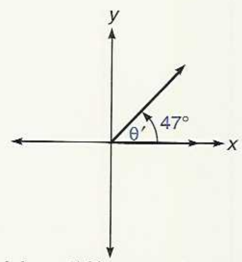


Figure 5–14

A reference angle is always acute (between 0° and 90°) *and is always formed by the terminal side of the angle and the x-axis* (never the y-axis). As will be illustrated in example 5–4 B, a good way to find a reference angle is to sketch the angle itself. This should make clear what computation to perform.

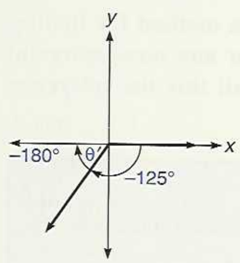
■ ***Example 5–4 B***

Compute and sketch the reference angle for each angle.



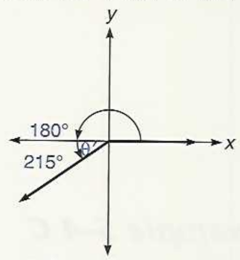
1. 47°

This angle terminates in quadrant I. The reference angle is the same as the angle itself, 47°.



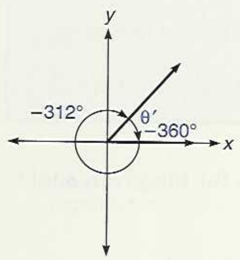
2. −125°

This angle terminates in quadrant III. The positive difference between −125° and −180° is 180° − 125° = 55°, which is the value of the reference angle.



3. 215°

This angle terminates in quadrant III also. Here the reference angle is 215° − 180° = 35°.



4. −312°

This angle terminates in quadrant I. The value of the reference angle is the positive difference between −312° and −360° = 360° − 312° = 48°. ■

It can be seen that if $0° < \theta < 360°$, then the reference angle θ' can be found according to the following formulas:

$$
\begin{aligned}
&\theta \text{ in quadrant I:} && \theta' = \theta \\
&\theta \text{ in quadrant II:} && \theta' = 180° - \theta \\
&\theta \text{ in quadrant III:} && \theta' = \theta - 180° \\
&\theta \text{ in quadrant IV:} && \theta' = 360° - \theta
\end{aligned}
$$

The absolute value of the trigonometric functions for any angle

The absolute value of a trigonometric function for any angle is the same as the trigonometric ratio for the corresponding reference angle. Figure 5–15 illustrates this idea for the angle 150°. If an angle of measure 150° is in standard position, then we find the values of the trigonometric functions by taking a point on its terminal side (the point $B(x,y)$ in the figure) and using the definitions of these functions in terms of x, y, and r.

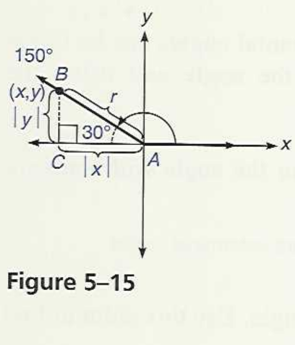


Figure 5–15

As seen in the figure, the absolute value of sin 150° is $\frac{|y|}{r}$. This is also the value of the trigonometric ratio for the reference angle, with measure 30°: $\sin 30° = \frac{\text{length of side opposite } 30°}{\text{length of hypotenuse}}$. We know from section 5–2 that $\sin 30° = \frac{1}{2}$. Thus, in absolute value, $|\sin 150°| = \sin 30° = \frac{1}{2}$.

The exact values of the trigonometric functions for certain angles

The facts discussed in the previous paragraphs provide a method for finding the *exact* values of the basic trigonometric functions for any nonquadrantal angle whose reference angle is 30°, 45°, or 60°. We call this the reference angle/ASTC procedure.

> **Reference angle/ASTC procedure**
>
> To find the value of a trigonometric function for a nonquadrantal angle whose reference angle is 30°, 45°, or 60°:
>
> 1. Find the value of the reference angle.
> 2. Find the value of the appropriate trigonometric ratio for the reference angle from table 5–1 in section 5–2.
> 3. Determine the sign of this value using the ASTC rule (figure 5–13).

■ ***Example 5–4 C***

Find the exact value of the given trigonometric function for the given angle.

1. cos 210°

$\theta' = 210° - 180° = 30°$ — Find the value of the reference angle

$\cos 30° = \dfrac{\sqrt{3}}{2}$ — Table 5–1 (memorized value)

$\cos 210° = -\dfrac{\sqrt{3}}{2}$ — A 210° angle terminates in quadrant III, where the cosine function is negative

2. tan 840°

$840° - 2(360°) = 120°$ — 120° and 840° are coterminal

$\theta' = 180° - 120° = 60°$ — Reference angle for 840°

$\tan 60° = \sqrt{3}$ — Memorized value

$\tan 840° = -\sqrt{3}$ — 840° terminates in quadrant II, where the tangent function is negative ■

The values of the trigonometric functions for quadrantal angles can be found by selecting any point on the terminal side of the angle and using the definitions.

■ ***Example 5–4 D***

Find the value of the six trigonometric functions for the angle with measure 900°.

$900° - 2(360°) = 180°$ — 900° and 180° are coterminal angles

Thus, sin 900° = sin 180°.
The point $(-1,0)$ is on the terminal side of a 180° angle. Use this point to find the values for 900°.

$$r = \sqrt{(-1)^2 + 0^2} = 1 \qquad x = -1, y = 0$$

$$\sin 900° = \frac{y}{r} = \frac{0}{1} = 0, \csc 900° = \frac{1}{\sin 900°} = \frac{1}{0}; \text{ undefined}$$

$$\cos 900° = \frac{x}{r} = \frac{-1}{1} = -1, \sec 900° = \frac{1}{\cos 900°} = \frac{1}{-1} = -1$$

$$\tan 900° = \frac{y}{x} = \frac{0}{-1} = 0, \cot 900° = \frac{1}{\tan 900°} = \frac{1}{0}; \text{ undefined}$$

Approximate values of the trigonometric functions—calculators

As with the trigonometric ratios it is difficult and in most cases impossible to find exact values for the trigonometric functions. The keys marked $\boxed{\sin}$, $\boxed{\cos}$, and $\boxed{\tan}$ calculate approximate values of these three trigonometric functions as necessary. It is not necessary to find a reference angle when using a calculator, since they are programmed to perform this step automatically.[5]

Recall from section 5–2 that the calculator must be in degree mode when entering angle measure in degrees. See example 5–2 A also.

Example 5–4 E

Find four decimal place approximations to the following function values.

1. sin 133° — 133 $\boxed{\sin}$ — Display $\boxed{0.731353701}$

sin 133° ≈ 0.7314 — TI-81: $\boxed{\text{SIN}}$ 133 $\boxed{\text{ENTER}}$

2. tan 62°14′ — 62 $\boxed{+}$ 14 $\boxed{\div}$ 60 $\boxed{=}$ $\boxed{\tan}$

tan 62°14′ ≈ 1.8993 — TI-81: $\boxed{\text{TAN}}$ 62 $\boxed{+}$ 14 $\boxed{\div}$ 60 $\boxed{\text{ENTER}}$

Display $\boxed{1.899346356}$

3. sec(−335.6°) — 335.6 $\boxed{\pm}$ $\boxed{\cos}$ $\boxed{1/x}$

sec(−335.6°) ≈ 1.0981 — TI-81: $\boxed{(}$ $\boxed{\text{COS}}$ $\boxed{(-)}$ 335.6 $\boxed{)}$ $\boxed{x^{-1}}$ $\boxed{\text{ENTER}}$

Display $\boxed{1.098076141}$

Solutions to trigonometric equations

Recall from section 5–1 that we use the inverse trigonometric functions to find the degree measure of an acute angle when given the value of one of the trigonometric ratios. The same idea can be used to solve trigonometric equations of the form $\sin\theta = k$, $\cos\theta = k$. and $\tan\theta = k$, where k is a known constant. In particular, to find one value of θ in each equation, we use the following facts.

[5]However, some older calculators have limits on how large the measure of an angle may be (for example, 1,000°).

if $\sin \theta = k$, then one solution for θ is $\theta = \sin^{-1}k$
if $\cos \theta = k$, then one solution for θ is $\theta = \cos^{-1}k$
if $\tan \theta = k$, then one solution for θ is $\theta = \tan^{-1}k$

In section 6–4 we will examine this situation in more depth, but for now we will simply rely on these facts, and on the fact that these inverse trigonometric functions are programmed into calculators.

Example 5–4 F

Find one solution to each trigonometric equation, to the nearest 0.1°.

1. $\sin \theta = -0.8500$

$\theta = \sin^{-1}(-0.8500) \approx -58.2°$ .85 $\boxed{+/-}$ $\boxed{\sin^{-1}}$

TI-81: $\boxed{\text{SIN}^{-1}}$ $\boxed{(-)}$ .85 $\boxed{\text{ENTER}}$

2. $\cos \theta = -0.2145$

$\theta = \cos^{-1}(-0.2145) \approx 102.4°$

Mastery points

Can you

- Determine in which quadrant an angle terminates when given the signs of two of the trigonometric function values for that angle?
- Compute and sketch the reference angle for a given nonquadrantal angle θ with given degree measure?
- Find the exact value of any trigonometric function for an angle whose reference angle is 30°, 45°, or 60°, using the reference angle/ASTC procedure?
- Find the exact value of any trigonometric function for a quadrantal angle?
- Find the approximate value of any trigonometric function using a calculator?
- Find the approximate value of one solution to an equation of the form $\sin \theta = k$, $\cos \theta = k$, $\tan \theta = k$?

Exercise 5–4

In the following problems you are given the sign of two of the trigonometric functions of an angle in standard position. State in which quadrant the angle terminates.

1. $\sin \theta > 0, \cos \theta < 0$
2. $\sec \theta < 0, \tan \theta > 0$
3. $\cos \theta > 0, \tan \theta > 0$
4. $\cot \theta < 0, \csc \theta > 0$
5. $\tan \theta < 0, \csc \theta < 0$
6. $\sec \theta > 0, \csc \theta < 0$
7. $\csc \theta > 0, \cos \theta < 0$
8. $\tan \theta > 0, \sin \theta < 0$
9. $\sec \theta > 0, \sin \theta < 0$
10. $\cot \theta > 0, \sin \theta > 0$
11. $\sin \theta < 0, \sec \theta < 0$

For each of the following angles, find the measure of the reference angle θ'.

12. 39.3° **13.** 164.2° **14.** 213.2° **15.** 427.1° **16.** −16.8°
17. −255.3° **18.** −100.4° **19.** 130.7° **20.** −671.3° **21.** −181.0°
22. 512.8° **23.** −279.5° **24.** 292.3° **25.** −252° **26.** 312°

Find the exact trigonometric function value for each angle.

27. sin 135° **28.** cos 120° **29.** sin 210° **30.** cos 330° **31.** tan 300°
32. sin 240° **33.** sin(−120°) **34.** cos(−315°) **35.** cos 660° **36.** csc(−315°)
37. cot 300° **38.** sin 450° **39.** cos(−450°) **40.** tan(−540°) **41.** csc 90°
42. sin 840° **43.** sin(−690°) **44.** cot 215° **45.** sec 150° **46.** tan 330°

Find the trigonometric function value for each angle to four decimal places.

47. sin 113.4° **48.** cos 88.2° **49.** tan 214.6° **50.** csc 345°10′ **51.** cot 412°
52. tan 527.2° **53.** sec(−13°) **54.** sin(−88°) **55.** cos(−355°20′) **56.** tan(−248.6°)
57. csc 285.3° **58.** sec 211° **59.** cos(−133.2°) **60.** sin(−293°50′)

Find one approximate solution to each equation, to the nearest 0.1°.

61. $\sin\theta = 0.25$ **62.** $\sin\theta = \frac{1}{3}$ **63.** $\cos\theta = -0.5$ **64.** $\cos\theta = 0.813$ **65.** $\tan\theta = -\frac{8}{5}$
66. $\tan\theta = 3$ **67.** $\sin\theta = -0.59$ **68.** $\cos\theta = -0.18$

69. In a certain electrical circuit the instantaneous voltage E (in volts) is found by the formula $E = 156\sin(\theta + 45°)$. Compute E to the nearest 0.01 volt for the following values of θ:
a. 0° **b.** 45° **c.** 100° **d.** −200° **e.** 13.3° **f.** −45°.

70. In a certain electrical circuit the instantaneous current I (in amperes) is found by the formula $I = 1.6\cos(800t)°$. Find I to the nearest 0.01 ampere for the following values of t:
a. 0 **b.** 0.25 **c.** 0.85 **d.** 1 **e.** −1 **f.** −2.5 **g.** −0.02.

71. If a force of 200 pounds is applied to a rope to drag an object, the actual force tending to move the object horizontally is $f(\theta) = 200\cos\theta$, where θ is the angle the rope makes with the horizontal. Compute the force tending to move the object horizontally if the angle of the rope is **a.** 0° **b.** 25° **c.** 50°

72. If a rocket is moving through the air at a speed of 1,200 mph, at an angle of $\theta°$ with the horizontal, then the rate at which it is rising is $v(\theta) = 1{,}200\sin\theta$. Find the rate at which a rocket moving at 1,200 mph is rising if the angle it makes with the horizontal is
a. 50° **b.** 60° **c.** 70° **d.** 80°

Skill and review

1. Use the values 30° and 60° to see if the statement $\sin(2\theta) = 2\sin\theta$ is true. (Let θ be 30°.)
2. Use the values 30° and 60° to see if the statement $\sin\frac{\theta}{2} = \frac{\sin\theta}{2}$ is true. (Let θ be 60°.)
3. Use the values 30°, 60°, 90° to see if the statement $\sin(\alpha + \beta) = \sin\alpha + \sin\beta$ is true.

5–5 Finding values from other values—reference triangles

A numerically controlled drill is being set up to drill a hole in a piece of steel 6.8 millimeters from the origin at an angle of 135°30′. What are the coordinates of this point?

The advent of numerically controlled machinery has made trigonometry more important than ever. The problem above is one of many situations where this is true.

Finding a general angle from a value and quadrant

In section 5–2 we learned how to find the degree measure of an acute angle if we know the value of one of the trigonometric ratios for that angle. We used the inverse sine, cosine, or tangent function as appropriate. We are now dealing with angles of any measure, but the same procedure can be used to find the value of a reference angle. From this we can find the least nonnegative measure for an angle.

As is illustrated in example 5–4 A we *always find a reference angle θ' by finding the inverse sine, cosine, or tangent function value for a positive value of x.* This is because the value of the sine, cosine, and tangent functions are positive for acute angles, and reference angles are acute angles. (The topic of inverse trigonometric functions is explored fully in section 6–4.) We could summarize the procedure as follows.

Finding the least nonnegative measure of an angle from a trigonometric function value and information about a quadrant

1. If necessary use the ASTC rule[6] to determine the quadrant for the terminal side of the angle.
2. Use $\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$ to find θ'. Use the absolute value of the given trigonometric function value.
3. Apply θ' to the correct quadrant to determine the value of θ.

Note We find the "least nonnegative value." There are actually an unlimited number of values, since the trigonometric values are the same for all coterminal angles.

In section 5–3 we saw formulas that find θ' if $0° < \theta < 360°$. These formulas can be solved for θ if necessary and thus provide a formula for finding θ given θ'.

Relationship between θ and θ' if $0° < \theta < 360°$

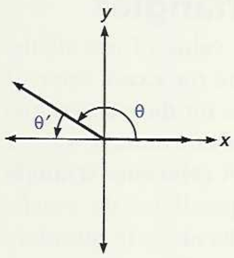
θ in quadrant I:	$\theta' = \theta$	$\theta = \theta'$
θ in quadrant II:	$\theta' = 180° - \theta$	$\theta = 180° - \theta'$
θ in quadrant III:	$\theta' = \theta - 180°$	$\theta = \theta' + 180°$
θ in quadrant IV:	$\theta' = 360° - \theta$	$\theta = 360° - \theta'$

It is interesting to observe that the formulas are the "same" when solved for θ and θ' for every quadrant except quadrant III.

[6] Section 5–4.

■ *Example 5–5 A*

Find the least nonnegative measure of θ to the nearest 0.1°.



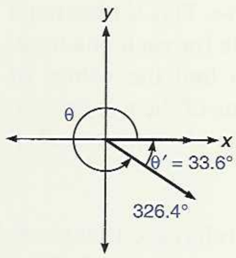
1. $\sin\theta = 0.5150$ and $\cos\theta < 0$

Since $\sin\theta > 0$ and $\cos\theta < 0$, θ terminates in quadrant II (see the figure). We find the acute reference angle θ′ just as we did in section 5–2.

$$\theta' = \sin^{-1}0.5150 \approx 31.0°$$

Thus, $\theta \approx 180° - 31.0° = 149.0°$.

Note The calculator can be used to verify our result by checking that $\sin 149° \approx 0.5150$ and that $\cos 149° < 0$.



2. $\tan\theta = -0.6644$ and $\sin\theta < 0$

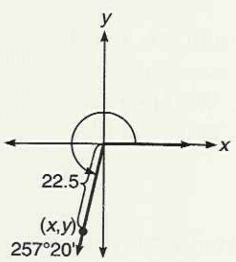
Since $\tan\theta < 0$ and $\sin\theta < 0$, θ terminates in quadrant IV.

$$\theta' = \tan^{-1}0.6644 \approx 33.6°$$ Note we use the positive value 0.6644

$$\theta = 360° - \theta' \approx 326.4°.$$ ■

There are many places in science and technology where we find applications for trigonometric functions. With the advent of numerically controlled, or computer-controlled, machines, these applications are becoming more common.

■ *Example 5–5 B*



A technician is setting up a numerically controlled grinding wheel. The starting position for the wheel must be at an angle of 257°20′ and must be 22.5 inches from the origin (assuming the machine uses our usual x-y coordinate system). Find the x- and y-coordinates of the point at which the grinding wheel must start, to the nearest tenth of an inch.

The figure illustrates the situation. We have $r = 22.5$ inches and $\theta = 257°20'$.

By definition, $\sin\theta = \frac{y}{r}$ and $\cos\theta = \frac{x}{r}$, so we find x and y as follows:

$$\sin 257°20' = \frac{y}{22.5}$$
$$y = 22.5 \sin 257°20'$$
$$y \approx -22.0 \text{ inches}$$

Note The calculation is

22.5 $\boxed{\times}$ $\boxed{(}$ 257 $\boxed{+}$ 20 $\boxed{\div}$ 60 $\boxed{)}$ $\boxed{\text{sin}}$ $\boxed{=}$ $\boxed{-21.9524013}$

TI-81: 22.5 $\boxed{\times}$ $\boxed{\text{SIN}}$ $\boxed{(}$ 257 $\boxed{+}$ 20 $\boxed{\div}$ 60 $\boxed{)}$ $\boxed{\text{ENTER}}$

$$\cos 257°20' = \frac{x}{22.5}$$
$$x = 22.5 \cos 257°20'$$
$$x \approx -4.9 \text{ inches}$$

Thus, the starting coordinates, in inches, for the grinder are (−4.9,−22.0). ■

Exact values of the trigonometric functions from a known value—reference triangles

There are many situations in which we know the exact value of one of the trigonometric functions for a given angle and need to find the exact value of one or more of the remaining five trigonometric functions for the same angle. We can do this by using a reference triangle, which is a convenient way of combining the idea of reference angle and right triangle. A **reference triangle** is a right triangle with one leg on the x-axis and one leg parallel to the y-axis. The acute angle on the x-axis is the reference angle for the angle in question. *The lengths of the legs of a reference triangle are treated as directed distances (i.e., positive or negative); the hypotenuse is always positive.* This is illustrated in example 5–5 C. Figure 5–16 shows a reference triangle for each quadrant.

The primary use of a reference triangle is to help find the values of the other five trigonometric functions when the value of one of them is known. A secondary use is to find the value of an angle. This is also illustrated in example 5–5 C.

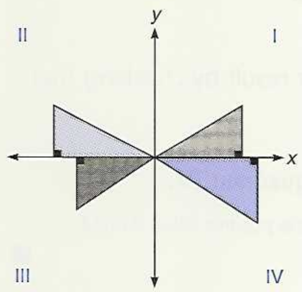
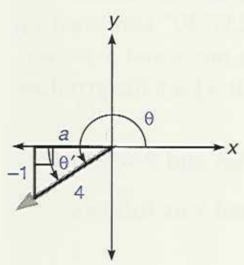


Figure 5–16

■ *Example 5–5 C*

In each case draw a representation of angle θ and use a reference triangle to help find the values of the other five trigonometric functions. Also, find the least positive value of θ to the nearest 0.1°.

1. $\sin\theta = -\dfrac{1}{4}$ and $\tan\theta > 0$



We know θ terminates in quadrant III since $\sin\theta < 0$ and $\tan\theta > 0$. We construct a right triangle in quadrant III in which one acute angle is a reference angle. This is shown in the figure. We label the hypotenuse 4 and the directed side opposite θ′ as −1. Thus, $\sin\theta' = \dfrac{\text{length of side opposite }\theta'}{\text{length of hypotenuse}} = -\dfrac{1}{4}$.

Note In a reference triangle the length of the hypotenuse is always positive.

$a^2 + (-1)^2 = 4^2$ Find the value of a using the Pythagorean theorem; since we are squaring values this theorem works for directed distances

$a^2 = 15$

$a = \pm\sqrt{15}$

We choose $a = -\sqrt{15}$ since it is negative as a directed distance.

We can now use the definitions of the trigonometric ratios for θ′ along with the directed distances to find the remaining trigonometric function values for θ.

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = -\frac{\sqrt{15}}{4},\quad \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{-1}{-\sqrt{15}} = \frac{\sqrt{15}}{15}$$

$$\csc\theta = \frac{1}{\sin\theta} = -4,\qquad \sec\theta = \frac{1}{\cos\theta} = -\frac{4}{\sqrt{15}} = -\frac{4\sqrt{15}}{15}$$

$$\cot\theta = \frac{1}{\tan\theta} = \sqrt{15}$$

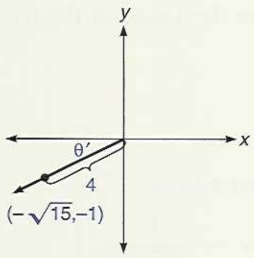


Figure 5–17

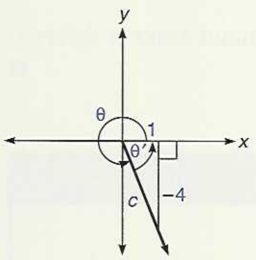
We now find an approximation to θ.

$$\sin\theta' = \tfrac{1}{4},\ \text{so}\ \theta' = \sin^{-1}\tfrac{1}{4} \approx 14.5°,\ \text{so}\ \theta \approx 180° + 14.5° = 194.5°.$$

Note The reference triangle works because it is equivalent to finding a point on the terminal side of θ and applying the definitions of the trigonometric functions (section 5–3). Finding the reference triangle in part 1 was equivalent to finding the point $(-\sqrt{15},-1)$ to be on the terminal side of angle θ. (See figure 5–17.)

2. $\cot\theta = -\frac{1}{4}$ and $270° < \theta < 360°$

If $\cot\theta = -\dfrac{1}{4}$ then $\tan\theta = -4 = -\dfrac{4}{1} = \dfrac{\text{opposite}}{\text{adjacent}}$. The fact that $270° < \theta < 360°$ means that θ is in quadrant IV. The figure shows a reference triangle for an angle in quadrant IV with tangent $\left(\dfrac{\text{opposite}}{\text{adjacent}}\right)$ of -4.



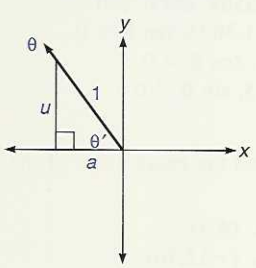
$$c^2 = 1^2 + (-4)^2$$
$$c = \sqrt{17}$$
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{-4}{\sqrt{17}} = -\frac{4\sqrt{17}}{17},\ \csc\theta = \frac{1}{\sin\theta} = -\frac{\sqrt{17}}{4}$$
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{17}} = \frac{\sqrt{17}}{17},\quad \sec\theta = \frac{1}{\cos\theta} = \sqrt{17}$$

$\tan\theta' = 4$, so $\theta' = \tan^{-1}4 \approx 76.0°$, and $\theta = 360° - \theta' \approx 284°$ ■

Note In section 5–1 we noted it is best to convert values of reciprocal trigonometric ratios to primary trigonometric ratios. This advice applies to the trigonometric functions as well, and it is why we first converted cot θ to tan θ in part 2 of example 5–5 C.

Example 5–5 D shows how to handle a case where the value of a trigonometric function is given in terms of a literal constant (i.e., a letter). This is in fact a situation often encountered in studying the Calculus.

■ *Example 5–5 D*



Draw a representation of angle θ and use a reference triangle to help find the values of the other five trigonometric functions if $\sin\theta = u$ and θ terminates in quadrant II.

In this problem the only given value is u. We therefore need to express the values of the trigonometric functions in terms which use only constants and the value u.

The figure shows a reference triangle in quadrant II, where $\sin\theta' = u$. Observe that we use the fact that $\dfrac{u}{1} = u$ to obtain u for the side of the triangle and 1 for the hypotenuse.

We need to find an expression for the length of the third side of the triangle, labeled a in the figure.

$$1 = u^2 + a^2 \quad \text{Pythagorean theorem}$$
$$1 - u^2 = a^2$$
$$\pm\sqrt{1 - u^2} = a$$
$$-\sqrt{1 - u^2} = a \quad \text{Choose } a < 0 \text{ as a directed distance}$$
$$\cos\theta = \frac{a}{1} = \frac{-\sqrt{1 - u^2}}{1} = -\sqrt{1 - u^2}, \sec\theta = -\frac{1}{\sqrt{1 - u^2}},$$
$$\tan\theta = -\frac{u}{\sqrt{1 - u^2}}, \cot\theta = -\frac{\sqrt{1 - u^2}}{u}, \csc\theta = \frac{1}{u}$$

Since we do not know the actual value of u we cannot make a determination of an approximate value for angle θ. ■

Mastery points

Can you

- Find an approximation to the least nonnegative measure of an angle, given the value of one of the trigonometric functions, and the sign of a second, for that angle?
- Apply the definitions of the trigonometric functions in appropriate situations?
- Use reference triangles to find the values of the remaining trigonometric functions for a given angle, when given the value of one of the trigonometric functions for that angle?

Exercise 5–5

Find the measure of the least nonnegative angle that meets the conditions given in the following problems, to the nearest 0.1°.

1. $\sin\theta = 0.8251, \cos\theta > 0$
2. $\cos\theta = -0.1771, \sin\theta < 0$
3. $\tan\theta = 0.6569, \sec\theta > 0$
4. $\sin\theta\ -0.6508, \tan\theta > 0$
5. $\sec\theta = -1.0642, \sin\theta < 0$
6. $\csc\theta = -1.3673, \tan\theta > 0$
7. $\tan\theta = -0.0349, \csc\theta < 0$
8. $\cos\theta = -0.2222, \sin\theta > 0$
9. $\sin\theta = \frac{3}{8}, \cos\theta > 0$
10. $\sin\theta = \frac{3}{8}, \cos\theta < 0$
11. $\cot\theta = -5, \sin\theta > 0$
12. $\tan\theta = -5, \sin\theta < 0$
13. $\cos\theta = -\frac{5}{7}, \tan\theta > 0$
14. $\cos\theta = -\frac{5}{7}, \tan\theta < 0$

In each problem you are given the coordinates of a point on the terminal side of θ. In each case (a) find the exact value of the trigonometric functions for θ and (b) find the least nonnegative measure of θ, to the nearest 0.1°.

15. (−3,4)
16. (−5,−12)
17. (12,−5)
18. (4,3)
19. (−4,−5)
20. (5,−4)
21. (4,6)
22. (−12,16)

In each case (a) draw a representation of angle θ, (b) use a reference triangle to help find the exact values of the three primary trigonometric functions (sine, cosine, tangent), and (c) find the least positive value of θ to the nearest $0.1°$.

23. $\sin\theta = \frac{3}{4}$, $\cos\theta > 0$
24. $\sin\theta = \frac{4}{5}$, $\cos\theta < 0$
25. $\cos\theta = -\frac{1}{2}$, $\tan\theta > 0$
26. $\cos\theta = -\frac{5}{13}$, $\tan\theta < 0$
27. $\sin\theta = 1$
28. $\cos\theta = 1$
29. $\tan\theta = 2$, $\cos\theta < 0$
30. $\tan\theta = 3$, $\cos\theta > 0$
31. $\csc\theta = -5$, $\sec\theta < 0$
32. $\csc\theta = -2$, $\sec\theta > 0$
33. $\csc\theta = -1$
34. $\sec\theta = -1$
35. $\sin\theta = -\frac{3}{4}$, $\tan\theta > 0$
36. $\sin\theta = -\frac{2}{5}$, $\tan\theta < 0$
37. $\sec\theta = 4$, $\csc\theta > 0$
38. $\sec\theta = \sqrt{6}$, $\csc\theta < 0$
39. $\cot\theta = \dfrac{\sqrt{2}}{3}$, $\sin\theta < 0$
40. $\cot\theta = \frac{1}{3}$, $\sin\theta > 0$
41. $\cos\theta = -\frac{5}{13}$, $\sin\theta > 0$
42. $\cos\theta = -\frac{12}{13}$, $\sin\theta < 0$
43. $\tan\theta = \frac{7}{2}$, $\sec\theta < 0$
44. $\tan\theta = \frac{7}{3}$, $\sec\theta > 0$
45. $\sec\theta = 5$, $\tan\theta > 0$
46. $\sec\theta = 4$, $\tan\theta < 0$
47. $\sin\theta = \dfrac{1}{\sqrt{5}}$, $\tan\theta < 0$
48. $\sin\theta = \dfrac{1}{\sqrt{3}}$, $\tan\theta > 0$

In problems 49–54, draw a representation of the angle θ and use a reference triangle to find values of the other five trigonometric functions in terms of u.

49. $\cos\theta = u$ and θ terminates in quadrant I.
50. $\tan\theta = u$ and θ terminates in quadrant I.
51. $\cos\theta = u$ and θ terminates in quadrant III.
52. $\tan\theta = u$ and θ terminates in quadrant III.
53. $\sin\theta = u + 1$ and θ terminates in quadrant I.
54. $\cos\theta = 1 - u$ and θ terminates in quadrant I.

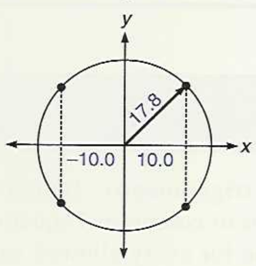
Solve the following problems.

55. A numerically controlled drill is being set up to drill a hole in a piece of steel 6.8 millimeters from the origin at an angle of 135°30′. To the nearest 0.01 millimeter, what are the coordinates of this point?

56. Suppose the hole in problem 55 must be 10.25 inches from the origin at an angle of 13°20′. Find the coordinates of this point to the nearest 0.01 inch.

57. Suppose the hole in problem 55 must be 8.25 centimeters from the origin at an angle of $-134.4°$. Find the coordinates of this point to the nearest 0.01 centimeter.

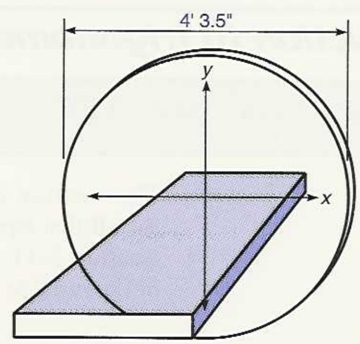
58. A numerically controlled drill must drill four holes on a circle whose center is at the origin with radius 17.8 centimeters, as shown in the diagram. The holes must be drilled wherever on this circle the x-coordinate is ± 10.0 centimeters. Find the y-coordinate and angle (to the nearest $0.1°$) for each of these four holes.



59. Suppose in problem 58 four additional holes must be drilled wherever the y-coordinate is ± 15.5 cm. Find the x-coordinate and angle for each of these holes.

60. A technician is aligning a laser device that is used to cut patterns out of cloth. The device is positioned at an angle of 135.20° and at a distance 5.50 feet from the origin. What should the x- and y-coordinates be at this point, to the nearest 0.01 foot?

61. A scanning device used in medical diagnosis has a moving part that moves with great precision in a circle around the patient. Assume the y-axis is perpendicular to the top of the table on which the patient lies and the x-axis is at right angles to the length of the table. The diameter of the machine is 4 feet 3.5 inches. Find the coordinates of the moving part when the angle is 211.5°, to the nearest 0.1 inch.



62. In the June 1980 issue of *Popular Science* magazine Mr. R. J. Ransil presented several formulas for calculating saw angles for compound miters. The formulas are:

$$\text{angle } A = 90° - \frac{180°}{\text{number of sides}}$$
$$\tan(\text{arm angle}) = \cot(A) \cdot \sin(\text{slope})$$
$$\sin(\text{tilt angle}) = \cos(A) \cdot \cos(\text{slope})$$

Arm angles and tilt angles are acute.

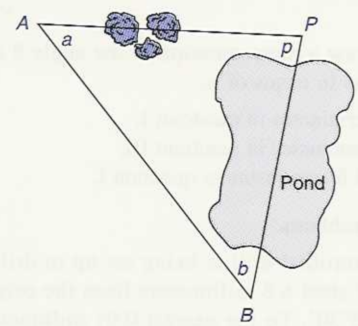
Calculate the arm angle and tilt angle to the nearest 0.1° for the following numbers of sides and slopes:

Number of sides	Slope (in degrees)
3	5
5	5
7	25
7	30
6	35
8	35

63. A surveying manual describes how to find distance BP in the figure. The distance AP can be found, but trees prevent measuring angle a. Angle b can be measured, but not distance BP. The manual instructs the surveyor to find BP by solving the following sequence of formulas:

$$\sin p = \frac{AB \sin b}{AP}$$
$$a = 180° - (b + p)$$
$$BP = \frac{AP \sin a}{\sin b}$$

Note that the first formula does not give angle p but only $\sin p$. Also, assume p is acute. Solve the sequence of formulas to compute the distance BP to the nearest 0.1 foot if $AB = 512.4$ feet, $AP = 322.6$ feet, and $b = 28.3°$.



64. Find BP in problem 63 to the nearest 0.1 meter if $AB = 319.2$ meters, $AP = 225.7$ meters, and $b = 31.6°$.

Skill and review

1. What is the reference angle for $-250°$?
2. The point $(3, -4)$ is on the terminal side of an angle in standard position. Find the value of the cosine of this angle.
3. In right triangle ABC, $A = 36.2°$, $c = 10.0$. Solve the triangle.
4. Use the graph of $y = x^3$ as a guide to graph $f(x) = (x + 1)^3 - 1$.
5. Solve the equation $\frac{3x - 5}{12} = 2(x - 3) - 8x$.
6. Solve the inequality $\frac{3x - 9}{x - 2} \le 0$.

5–6 Introduction to trigonometric equations

Solve the equation $2\cos^2\theta - \cos\theta - 1 = 0$.

This section introduces equations involving the trigonometric functions. Recall that equations can be categorized as identities or conditional equations (section 2–1). An *identity* is an equation that is true for every allowed value of its variable (or variables). For example, $2(x + 3) = 2x + 6$ is an identity,

since the left member and right member of the equation represent the same value, regardless of the value of x. Similarly, $\frac{3x^2}{3x} = x, \quad x \neq 0$ is an identity.

A *conditional equation* is an equation that is true only for some, but not all, values that may replace the variable. For example, $6x = 12$ is true only if x is replaced by 2, and $x^2 = 9$ is true only if x is replaced by 3 or -3.

Identities

We have seen the following identities

$$[1] \qquad \csc\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta}$$

$$[2] \qquad \sin\theta = \frac{1}{\csc\theta}, \quad \cos\theta = \frac{1}{\sec\theta}, \quad \tan\theta = \frac{1}{\cot\theta}$$

Knowing these identities permits us to simplify certain trigonometric expressions.

■ ***Example 5–6 A***

Simplify each trigonometric expression.

1. $\csc\theta\sin\theta = \frac{1}{\sin\theta}\cdot\sin\theta$ — Replace $\csc\theta$ by $\frac{1}{\sin\theta}$

$= 1$ — $\frac{1}{a}\cdot a = 1$

2. $\frac{1-\csc\theta}{\csc\theta} = \frac{1-\csc\theta}{\csc\theta}$

$= \frac{1}{\csc\theta} - \frac{\csc\theta}{\csc\theta}$ — $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$

$= \sin\theta - 1$ — $\frac{1}{\csc\theta} = \sin\theta$ and $\frac{\csc\theta}{\csc\theta} = 1$ ■

Two more useful identities we have seen are

$$[3] \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

■ ***Example 5–6 B***

Simplify the expression.

$\cot\alpha(\sin\alpha - \tan\alpha)$

$\cot\alpha\sin\alpha - \cot\alpha\tan\alpha$ — $a(b+c) = ab + ac$

$\frac{\cos\alpha}{\sin\alpha}\sin\alpha - \frac{1}{\tan\alpha}\tan\alpha$ — $\cot\theta = \frac{\cos\theta}{\sin\theta}$, $\cot\theta = \frac{1}{\tan\theta}$

$\cos\alpha - 1$ — Reduce common numerators and denominators

Note We replaced $\cot\alpha$ by $\frac{\cos\alpha}{\sin\alpha}$ in one term, and by $\frac{1}{\tan\alpha}$ in another term. We use whichever identity better suits the rest of the term. ■

Another very important identity is often called the fundamental identity of trigonometry.

> **Fundamental identity of trigonometry**
> For any angle θ, $\sin^2\theta + \cos^2\theta = 1$.

The term $\sin^2\theta$ means $(\sin\theta)^2$, and $\cos^2\theta$ means $(\cos\theta)^2$. This identity can be shown to be true as follows:

$$
\begin{aligned}
\sin^2\theta + \cos^2\theta &= (\sin\theta)^2 + (\cos\theta)^2 \\
&= \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 && \sin\theta = \frac{y}{r},\ \cos\theta = \frac{x}{r} \\
&= \frac{x^2 + y^2}{r^2} \\
&= \frac{r^2}{r^2} = 1 && \text{Recall that } r^2 = x^2 + y^2
\end{aligned}
$$

Example 5–6 C illustrates applying the fundamental identity of trigonometry.

■ ***Example 5–6 C***

1. Simplify the expression.

$$
\begin{aligned}
&\left(\frac{1}{\sec\theta}\right)^2 + \left(\frac{1}{\csc\theta}\right)^2 \\
&(\cos\theta)^2 + (\sin\theta)^2 \\
&\cos^2\theta + \sin^2\theta \\
&1
\end{aligned}
$$

2. Simplify the expression.

$$
\begin{aligned}
&1 - \sin^2\beta \\
&(\sin^2\beta + \cos^2\beta) - \sin^2\beta && \text{Replace 1 by } \sin^2\beta + \cos^2\beta \\
&\cos^2\beta && \sin^2\beta - \sin^2\beta = 0
\end{aligned}
$$

3. Verify the fundamental identity of trigonometry for $\theta = 60°$.

$$
\begin{aligned}
\sin^2 60° + \cos^2 60° &= \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \\
&= \tfrac{3}{4} + \tfrac{1}{4} = 1
\end{aligned}
$$

■

Conditional trigonometric equations

We have already seen how to solve simple equations like $\tan\theta = 4$ by using the inverse tangent function to obtain one value of θ ($\tan^{-1}4 \approx 76°$, sections 5–2 and 5–4). There are many other solutions to this equation (in fact an unlimited number), which includes, among others, all angles coterminal to

Quadrant for least nonnegative solution

Function	Function value is positive	Function value is negative
Sine	I	III
Cosine	I	II
Tangent	I	II

Table 5–2

76°. This will be explored in more detail in section 7–4; *here we will restrict ourselves to the least nonnegative solution* to trigonometric equations. Considering the ASTC rule will show that the least nonnegative angle for which a trigonometric function value is positive is in quadrant I. When the function has a negative value ($\sin\theta = -\frac{1}{2}$ for example) the least nonnegative value is in quadrant II for the cosine and tangent function, and quadrant III for the sine function. This is summarized in table 5–2.

As we solve these equations whenever we compute an inverse trigonometric function to solve an equation we use the absolute value of the argument. This gives us the reference angle of the answer (as previously discussed in example 5–5 A). This is illustrated in parts 3 and 4 of example 5–6 D and can be summarized as

$$\text{if } \sin\theta = k, \text{ then } \theta' = \sin^{-1}|k|$$
$$\text{if } \cos\theta = k, \text{ then } \theta' = \cos^{-1}|k|$$
$$\text{if } \tan\theta = k, \text{ then } \theta' = \tan^{-1}|k|$$

■ Example 5–6 D

Solve the following conditional equations for the least nonnegative solution to the nearest 0.1°.

1. $2\sin x = \sqrt{3}$

$2\sin x = \sqrt{3}$

$\sin x = \dfrac{\sqrt{3}}{2}$ — Divide both members by 2

$x' = \sin^{-1}\dfrac{\sqrt{3}}{2}$ — If $\sin\theta = k$, then $\theta' = \sin^{-1}|k|$

$x' = 60°$

$x = 60°$ — $\sin x > 0$, so x is in quadrant I or II; the least nonnegative solution would be in quadrant I

2. $4\cos 3\alpha = 3$

$4\cos 3\alpha = 3$

$\cos 3\alpha = \frac{3}{4}$ — Divide both members by 4

$(3\alpha)' = \cos^{-1}\frac{3}{4}$ — If $\cos\theta = k$, then $k' = \cos^{-1}|k|$

$(3\alpha)' \approx 41.41°$ — .75 $\boxed{\cos^{-1}}$ Display $\boxed{41.40962211}$

TI-81: $\boxed{\text{COS}^{-1}}$ .75 $\boxed{\text{ENTER}}$

$3\alpha \approx 41.41°$ — Since $\cos 3\alpha > 0$ the least nonnegative value of 3α is in quadrant I; in quadrant I, the reference angle and the angle itself are coterminal

$\alpha \approx 13.8°$ — Divide both members by 3; round the result

3. $\tan \frac{x}{2} = -1.5$

$\tan \frac{x}{2} = -1.5$

$\frac{x'}{2} = \tan^{-1}1.5 \approx 56.31°$ — If $\tan \theta = k$, then $k' = \tan^{-1}|k|$

$\frac{x}{2} \approx 180° - 56.31°$ — $\frac{x}{2}$ terminates in quadrant II, since $\tan \frac{x}{2} < 0$ and we want the least nonnegative value of x; thus, $\theta = 180° - \theta'$

$\approx 123.69°$

$x \approx 247.4°$ — $2(123.69) \approx 247.4$

4. $2 \sin^2\theta - \sin \theta - 1 = 0$

This equation is quadratic in the variable $\sin \theta$. It can be factored. If this is difficult to see, use substitution for expression (section 1–3) as follows:

$2 \sin^2\theta - \sin \theta - 1 = 0$

$2u^2 - u - 1 = 0$ — Replace $\sin \theta$ by u, so $\sin^2\theta$ is replaced by u^2

$(2u + 1)(u - 1) = 0$

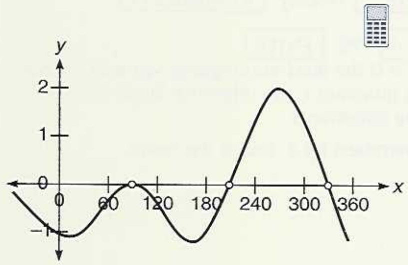
$(2 \sin \theta + 1)(\sin \theta - 1) = 0$ — Replace u by $\sin \theta$

$2 \sin \theta + 1 = 0$ or $\sin \theta - 1 = 0$ — Zero product property

$2 \sin \theta = -1$	$\sin \theta = 1$
$\sin \theta = -\frac{1}{2}$	
θ in quadrant III	
$\theta' = \sin^{-1}\frac{1}{2}$	$\theta' = \sin^{-1}1$
$\theta' = 30°$	$\theta' = 90°$
$\theta = 210°$	$\theta = 90°$

Thus, there are two solutions, 90° and 210°.

Note a. We took the least nonnegative solution for *each* factor.
b. The steps involving the substitution by u can be skipped if you see that $2 \sin^2\theta - \sin \theta - 1$ factors into $(2 \sin \theta + 1)(\sin \theta - 1)$.



Using a graphing calculator it is possible to visualize the solutions to these equations. The graph below is that of $y = 2 \sin^2\theta - \sin \theta - 1$. Where the graph crosses the x-axis, y is 0, and therefore the points at which the graph crosses the x-axis represent solutions to the equation $2 \sin^2\theta - \sin \theta - 1 = 0$.

The graph shows the two solutions we obtained in part 4 of example 5–6 D (at 90° and at 210°). A third solution occurs at 330°. We did not find this algebraically because we are taking the least nonnegative solution to each part of the equation. Chapter 7 will show how to find all solutions to trigonometric equations. To obtain this graph on the TI-81, make sure the calculator is in degree mode, and use the RANGE values Xmin=−30, Xmax=360, Ymin=−1, Ymax=2, Xscl=60. Enter the equation as $\boxed{Y=}$ 2 $\boxed{(}$ $\boxed{SIN}$ $\boxed{X|T}$ $\boxed{)}$ $\boxed{x^2}$ $\boxed{-}$ $\boxed{SIN}$ $\boxed{X|T}$ $\boxed{-}$ 1.

Mastery points

Can you

- Simplify simple trigonometric expressions?
- Solve simple equations involving the trigonometric functions?

Exercise 5–6

Show that each of the expressions in the left member can be transformed into the expression in the right member.

1. $\tan\theta\cot\theta = 1$

2. $\sec\theta\cos\theta = 1$

3. $\cos\theta(1 - \sec\theta) = \cos\theta - 1$

4. $\cot\alpha(\tan\alpha + \sin\alpha) = 1 + \cos\alpha$

5. $\sec\theta(\cot\theta + \cos\theta - 1) = \csc\theta - \sec\theta + 1$

6. $\dfrac{\cos\theta - 1}{\sin\theta} = \cot\theta - \csc\theta$

7. $\dfrac{\cos\alpha - \sin\alpha}{\cos\alpha} = 1 - \tan\alpha$

8. $\dfrac{\sin\theta + \cos\theta - 2}{\cos\theta} = \tan\theta + 1 - 2\sec\theta$

9. $1 - \cos^2\theta = \sin^2\theta$

10. $\cos\theta\cos\theta + \sin^2\theta = 1$

11. $\cos\beta(\sec\beta - \cos\beta) = \sin^2\beta$

12. $-\sin\theta(\sin\theta - \csc\theta) = \cos^2\theta$

13. $(\cos\theta + \sin\theta)(\cos\theta - \sin\theta) + 2\sin^2\theta = 1$

14. $\tan x(\cot x + \csc x) = 1 + \sec x$

15. $\csc\alpha(\cos\alpha - \sin\alpha) = \cot\alpha - 1$

16. $\sin\beta(\cot\beta - \csc\beta + \sin\beta) = \cos\beta - \cos^2\beta$

17. $\dfrac{\sin x - \cos x}{\sin x} = 1 - \cot x$

18. $\cos\alpha(\csc\alpha + \sec\alpha) = \cot\alpha + 1$

19. $\tan\beta(\cot\beta - \cos\beta) = 1 - \sin\beta$

20. Verify by computation that the fundamental identity is true when $\theta = 30°$.

21. Using approximate values check the fundamental identity when
a. $\theta = 16°50'$ **b.** $\theta = 50°$

22. Use the two identities $\cot\theta = \dfrac{1}{\tan\theta}$ and $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ to show that $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$.

23. Use approximate values to show that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ when $\theta = 32°40'$.

24. Show that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ when $\theta = 60°$ (use values from table 5–1).

Solve the following conditional equations for the least nonnegative solution to the nearest 0.1°.

25. $2\cos x = 1$

26. $\sqrt{3}\tan x = 1$

27. $2\sin x = \sqrt{3}$

28. $\sqrt{2}\cos x = 1$

29. $5\sin x = 1$

30. $3\sin x = 2$

31. $2\tan x = 9$

32. $4\cot x = 3$

33. $\dfrac{\sin x}{3} = \dfrac{2}{11}$

34. $\dfrac{\csc x}{3} = 2$

35. $\sin 3x = \frac{1}{2}$

36. $\cos 2x = \frac{1}{2}$

37. $\tan 2x = \sqrt{3}$

38. $\sin 2x = 0.8$

39. $\csc 3x = 3$

40. $4\sin 2x = 3$

41. $2\cos 4x = -1$

42. $-3\sin 2x = 0.75$

43. $-2\tan 3x = 8$

44. $\frac{1}{2}\sin 3x = -\frac{1}{4}$

45. $4\sin^2\theta - 1 = 0$

46. $2\sin^2\theta + \sin\theta - 1 = 0$

47. $2\cos^2\theta - \cos\theta - 1 = 0$

48. $4\sin^2\theta - 4\sin\theta + 1 = 0$

49. $\cos^2\theta - 1 = 0$

50. $2\sin^2\theta - 1 = 0$

51. $4\tan^2\theta + 4\tan\theta + 1 = 0$

Skill and review

1. $\cos\theta = -\frac{1}{3}$ and $\tan\theta < 0$. **a.** Use a reference angle to find the exact value of $\tan\theta$. **b.** Find the value of θ to the nearest 0.1°.
2. A 28 foot ladder is leaning against a building so that its base is 5 feet from the base of the building. Find the measure of the acute angle that the ladder makes with the ground, to the nearest 0.1°.
3. The circumference C of a circle is 28.5 feet. Use $C = 2\pi r$ to find the radius r to the nearest 0.1 feet.
4. Solve the equation $7x^2 + 14x - 10 = 6x^2 + 12x + 5$.

Chapter 5 summary

- $1° = 60'$ (one degree is equivalent to 60 minutes)
  $1' = 60''$ (one minute is equivalent to 60 seconds)
  $1° = 3600''$ (one degree is equivalent to 3,600 seconds)
- The sum of the measures of the angles of a triangle is 180°.
- A **right triangle** is a triangle in which one of the angles is a right (90°) angle. The side of a right triangle opposite the right angle is called the **hypotenuse.** This is always the longest side of the triangle.
- **The Pythagorean theorem** In a right triangle with hypotenuse of length c and sides of lengths a and b, $a^2 + b^2 = c^2$.
- **The primary trigonometric ratios** If θ is either of the two acute angles in a right triangle, then

$$\sin\theta = \frac{\text{length of side opposite } \theta}{\text{length of hypotenuse}}$$

$$\cos\theta = \frac{\text{length of side adjacent to } \theta}{\text{length of hypotenuse}}$$

$$\tan\theta = \frac{\text{length of side opposite } \theta}{\text{length of side adjacent to } \theta}.$$

- **Reciprocal pair ratios** If θ is either acute angle of a right triangle, $\csc\theta = \dfrac{1}{\sin\theta}$, $\sec\theta = \dfrac{1}{\cos\theta}$, $\cot\theta = \dfrac{1}{\tan\theta}$.
- **Important relations**

$$\sin\theta = \frac{1}{\csc\theta},\ \cos\theta = \frac{1}{\sec\theta},\ \tan\theta = \frac{1}{\cot\theta}.$$

- An **angle in standard position** is formed by two rays with the vertex at the origin. The **initial side** of the angle lies on the nonnegative portion of the x-axis. The **terminal side** of the angle may be in any quadrant or along any axis.
- **Coterminal angles** Two angles are coterminal if the measure of one can be formed from the measure of the other by adding or subtracting multiples of 360°.
- **The trigonometric functions** Let θ be an angle in standard position, and let (x,y) be any point on the terminal side of angle θ, except $(0,0)$. Let $r = \sqrt{x^2 + y^2}$ be the distance from the origin to the point. Then,

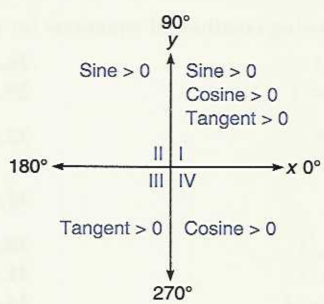
$$\sin\theta = \frac{y}{r},\quad \cos\theta = \frac{x}{r},\quad \tan\theta = \frac{y}{x}$$

$$\csc\theta = \frac{r}{y},\quad \sec\theta = \frac{r}{x},\quad \cot\theta = \frac{x}{y}$$

- **Tangent/cotangent identities**

$$\tan\theta = \frac{\sin\theta}{\cos\theta},\quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

- **Quadrantal angles** have degree measures which are integer multiples of 90°: 0°, ±90°, ±180°, ±270°, etc.
- **The ASTC rule** is illustrated in the figure. This rule gives the sign of the value of a trigonometric ratio for a nonquadrantal angle, based on the quadrant in which the angle terminates.



- A **reference angle** for a nonquadrantal angle is the acute angle formed by the terminal side of the angle and the x-axis.
- If $0° < \theta < 360°$ then the reference angle θ' can be found according to the following formulas:

θ in quadrant I: $\theta' = \theta$
θ in quadrant II: $\theta' = 180° - \theta$
θ in quadrant III: $\theta' = \theta - 180°$
θ in quadrant IV: $\theta' = 360° - \theta$

- **Values of sine, cosine, and tangent for special angles**

	Sine	Cosine	Tangent
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

- **Finding a general angle from a value and quadrant** Find a reference angle θ' by finding the inverse sine, cosine, or tangent function value for a *positive* value of x. Then use the ASTC rule to establish the actual quadrant of the angle θ, according to the following rules:

θ in quadrant I: $\theta = \theta'$
θ in quadrant II: $\theta = 180° - \theta'$
θ in quadrant III: $\theta = 180° + \theta'$
θ in quadrant IV: $\theta = 360° - \theta'$

- **Fundamental identity of trigonometry** For any angle θ,

$$\sin^2\theta + \cos^2\theta = 1$$

- **Solving conditional trigonometric equations**

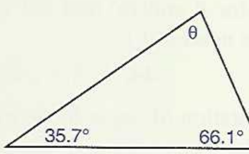
if $\sin\theta = k$, then $\theta' = \sin^{-1}|k|$
if $\cos\theta = k$, then $\theta' = \cos^{-1}|k|$
if $\tan\theta = k$, then $\theta' = \tan^{-1}|k|$

Chapter 5 review

[5–1] Convert each angle to its measure in decimal degrees. Round the answer to the nearest 0.001° where necessary. Also, state whether each angle is acute or obtuse.

1. 165°47′ **2.** 37°18′

3. Find the measure of angle θ in the figure.



In the following problems find the length of the missing side of right triangle ABC. Leave your answer in both exact form (in terms of rational numbers and radicals) and decimal form, to the nearest tenth.

	a	b	c		a	b	c
4.	3	5	?	**5.**	10	?	20
6.	?	$2\sqrt{2}$	10	**7.**	5	12	?

8. A ladder is 24 feet long, and is leaning against a building so its base is 4.5 feet from the base of the building. How far up the building does the ladder reach?

9. Find the ground speed, to the nearest knot, of an aircraft flying with a heading due west and an airspeed of 120 knots, if there is a wind blowing from the north at 16 knots.

[5–2] Use a calculator to find four-decimal-place approximations for the following.

10. sin 48.3° **11.** cot 58.7°
12. tan 10°20′ **13.** sec 4°38′

14. In the mathematical modeling of an aerodynamics problem the following equation arises:

$$y = x\cos A\cos B - x^2\cos A\sin B - x^3\sin A$$

Compute y to two decimal places if $x = 3.0$, $A = 13.2°$, and $B = 6°$.

Find the unknown acute angle θ to the nearest 0.01°.

15. $\sin\theta = 0.215$ **16.** $\sec\theta = 1.1028$

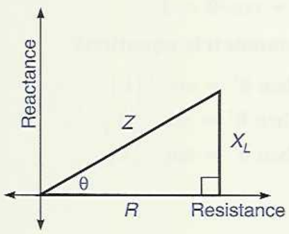
In the following problems you are given one side and one angle of a right triangle. Solve the triangle. Round all answers to the same number of decimal places as the data.

17. $a = 12.1$, $B = 30.3°$ **18.** $c = 12.6$, $A = 21.9°$

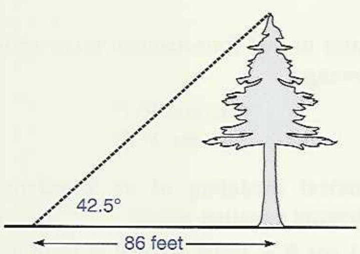
In the following problems you are given two sides of a right triangle. Solve the triangle. Round all lengths to the same number of decimal places as the data and all angles to the nearest 0.1°.

19. $a = 25.1$, $b = 15.0$ **20.** $b = 5.67$, $c = 10.40$

21. The figure illustrates an impedance diagram used in electronics theory. If Z (impedance) = 60.0 ohms and X_L (inductive reactance) = 25.0 ohms, find θ (phase angle) to the nearest degree and R (resistance) to the nearest 0.01 ohm.



22. An angle of elevation is an angle formed by one horizontal ray and another ray that is above the horizontal. The figure shows measurements made to find the height of a tree. Find the height to the nearest foot, if the angle of elevation is 42.5°, made 86 feet from the foot of the tree.



[5–3] In the following exercises state the measure of the smallest nonnegative angle that is coterminal with the given angle.

23. 480° **24.** −140° **25.** 1,256° **26.** −28°45′

[5–4] In the following problems you are given the sign of two of the trigonometric functions of an angle in standard position. State in which quadrant the angle terminates.

27. $\cot\theta > 0$, $\cos\theta < 0$ **28.** $\csc\theta < 0$, $\cot\theta < 0$

For each of the following angles, find the measure of the reference angle θ'.

29. 152.6° **30.** −172.3° **31.** −13.22°
32. 429.40° **33.** −250°

Find the trigonometric function value for each angle. If the reference angle is 30°, 45°, or 60°, find the exact trigonometric function value; otherwise find the required value to four decimal places.

34. sin 135° **35.** cos 48.5° **36.** tan 120°
37. csc 300° **38.** cot(−19°45′)

[5–5]

39. For a surveyor to locate a point by measuring an angle at one station and a distance from another one, the distance BP must be found by solving the following sequence of formulas:

$$\sin p = \frac{AB \sin b}{AP}$$
$$a = 180° - (b + p)$$
$$BP = \frac{AP \sin a}{\sin b}$$

Assume p is acute. Compute the distance BP to the nearest 0.1 foot if $AB = 420$ feet, $AP = 410$ feet, and $b = 20°$.

Find the measure of the least nonnegative angle that meets the conditions given in the following problems, to the nearest 0.1°.

40. $\cos\theta = 0.5252$, $\sin\theta > 0$
41. $\sin\theta = -0.8133$, $\tan\theta > 0$

In each problem you are given the coordinates of a point on the terminal side of θ. In each case (a) find the exact value of the trigonometric functions for θ and (b) find the least nonnegative measure of θ, to the nearest 0.1°.

42. (2,−6) **43.** (−2,4) **44.** $(-5,-\sqrt{2})$

In each case (a) draw a representation of angle θ, (b) use a reference triangle to help find the exact values of the other three primary trigonometric functions (sine, cosine, tangent), and (c) find the least positive value of θ to the nearest 0.1°.

45. $\sin\theta = \frac{4}{7}$, $\cos\theta > 0$ **46.** $\tan\theta = -4$, $\cos\theta > 0$
47. $\csc\theta = \sqrt{3}$, $\sec\theta < 0$

Find values of the three primary trigonometric functions in terms of u.

48. $\cos\theta = u$ and θ terminates in quadrant II.
49. $\tan\theta = u$ and θ terminates in quadrant II.

[5–6] Simplify the following trigonometric expressions.

50. $\sin\theta \csc\theta$ **51.** $\sec\alpha(\cos\alpha - \cot\alpha)$

52. $\dfrac{\sin\theta + 1}{\sin\theta}$ **53.** $\dfrac{\sin\beta - 1}{\cos\beta}$

54. $\cos\theta(\sec\theta - \cos\theta)$

Show that the following are identities.

55. $\cot x\left(\sec x - \tan x + \dfrac{1}{\cot^2 x}\right) = \csc x - 1 + \tan x$

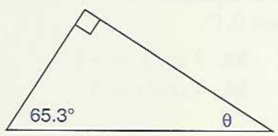
56. $(\sin \alpha - \cos \alpha)(\csc \alpha + \sec \alpha) = \tan \alpha - \cot \alpha$

Solve the following conditional equations for the least nonnegative solution to the nearest 0.1°.

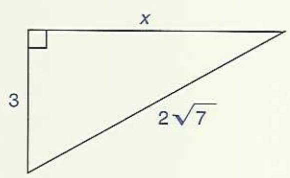
57. $2 \sin x = 1$ **58.** $2 \cos x = -\sqrt{3}$ **59.** $3 \tan x = 5$
60. $\sin 2x = 0.8$ **61.** $\sec 3x = -2$ **62.** $5 \sin 2x = 3$
63. $2 \cos^2\theta + \cos \theta - 1 = 0$

Chapter 5 test

1. Find the measure of angle θ in the figure.



2. Find the length x in the triangle in the figure.



3. A tree that is 46 feet tall has fallen against a building. It reaches 38 feet up a building. How far from the base of the building is the base of the leaning tree, to the nearest foot?

4. Find the ground speed, to the nearest knot, of an aircraft flying with a heading due south and an airspeed of 260 knots, if there is a wind blowing from the west at 30 knots.

Use a calculator to find four-decimal-place approximations for the following.

5. sin 27.3° **6.** tan 11.0°
7. cot 38°10′ **8.** sec 41°38′

9. In the mathematical modeling of an aerodynamics problem the following equation arises:

$$y = x \cos A \cos B - x^2 \cos A \sin B - x^3 \sin A$$

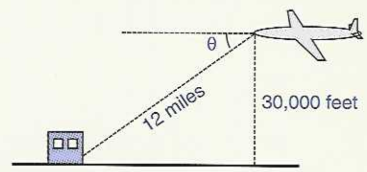
Compute y to two decimal places if $x = 1.8$, $A = 6.2°$, and $B = 21°$.

10. If $\cos \theta = 0.8711$, find θ to the nearest 0.1°.

11. In right triangle ABC, $b = 83$ and $B = 19.3°$. Solve the triangle. Leave answers to the nearest 0.1.

12. In right triangle ABC, $b = 83$ and $c = 125$. Solve the triangle. Leave answers to the nearest 0.1.

13. An *angle of depression* is an angle formed by one horizontal ray and another ray that is below the horizontal. The figure shows the situation of an aircraft flying at 30,000 feet, whose radar measures a slant distance of 12 miles to a building. Find the angle of depression of the radar beam (θ). Remember that one mile is 5,280 feet.



14. Find the least nonnegative angle that is coterminal with an angle of measure 675°.

15. If $\csc \theta > 0$, $\cos \theta < 0$, determine in which quadrant θ terminates.

For each of the following angles, find the measure of the reference angle θ′.

16. 211° **17.** −192.1°

Find the trigonometric function value for each angle. If the reference angle is 30°, 45°, or 60°, find the exact trigonometric function value; otherwise find the required value to four decimal places.

18. cos 219.8° **19.** sec 315°
20. cot 128° **21.** sec 310°

22. For a surveyor to locate a point by measuring an angle at one station and a distance from another one, the distance BP must be found by solving the following sequence of formulas: $\sin p = \dfrac{AB \sin b}{AP}$; $a = 180° - (b + p)$; $BP = \dfrac{AP \sin a}{\sin b}$. Assume p is acute. Compute the distance BP to the nearest 0.1 meter if $AB = 210$ meters, $AP = 150$ meters, and $b = 15°$.

23. Find the measure of the least nonnegative angle θ for which $\sin \theta = -0.4$ and $\tan \theta < 0$, to the nearest 0.1°.

In each problem you are given the coordinates of a point on the terminal side of θ. In each case (a) find the exact value of the trigonometric functions for θ and (b) find the least nonnegative measure of θ, to the nearest $0.1°$.

24. $(12, -6)$ **25.** $(-5, \sqrt{5})$

26. $\csc\theta = -\frac{4}{3}$, $\cos\theta > 0$. (a) draw a representation of angle θ, (b) use a reference triangle to help find the exact values of the three primary trigonometric functions (sine, cosine, tangent), and (c) find the least positive value of θ to the nearest $0.1°$.

27. If $\tan\theta = \dfrac{u}{2}$ and θ terminates in quadrant II, find the values of the other five trigonometric functions in terms of u.

Simplify the following trigonometric expressions.

28. $\tan\theta\cot\theta$ **29.** $\sec\theta(\cos\theta - \cos^3\theta)$

30. Show that $(\sin\theta + \cos\theta)^2 - \sin\theta\cos\theta = 1 + \sin\theta\cos\theta$ is an identity.

Solve the following conditional equations for the least nonnegative solution to the nearest $0.1°$.

31. $2\sin x = \sqrt{2}$ **32.** $3\tan x = -5$

33. $\cos 2x = 0.62$ **34.** $4\cos^2 x = 3$

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